

Spherical functions on the space of p -adic unitary hermitian matrices

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ABSTRACT

We investigate the space X of unitary hermitian matrices over \mathfrak{p} -adic fields through spherical functions on X . First we consider Cartan decomposition of X , and give precise representatives for fields with odd residual characteristic, i.e., $2 \notin \mathfrak{p}$. In the latter half we assume odd residual characteristic, and give explicit formulas of typical spherical functions on X , where Hall-Littlewood symmetric polynomials of type C_n appear as a main term, parametrization of all the spherical functions. By spherical Fourier transform, we show the Schwartz space $\mathcal{S}(K \backslash X)$ is a free Hecke algebra $\mathcal{H}(G, K)$ -module of rank 2^n , where n is the size of matrices in X , and give the explicit Plancherel formula on $\mathcal{S}(K \backslash X)$.

0. Introduction

Let k' be an unramified quadratic extension of a \mathfrak{p} -adic field k , and we consider hermitian and unitary matrices with respect to k'/k . For a matrix $A = (a_{ij}) \in M_{mn}(k')$, we denote by $A^* \in M_{nm}(k')$ the conjugate transpose with respect to k'/k , and say A is *hermitian* if $A^* = A$. We introduce the unitary group and the space of unitary hermitian matrices:

$$G = U(j_{2n}) = \{ g \in GL_{2n}(k') \mid g^* j_{2n} g = j_{2n} \}, \quad j_{2n} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in GL_{2n}(k'),$$

$$X = \{ x \in G \mid x^* = x, \Phi_{xj_{2n}}(t) = (t^2 - 1)^n \},$$

where $\Phi_y(t)$ is the characteristic function of the matrix y . The group G acts on X by

$$g \cdot x = gxg^*, \quad (g \in G, x \in X).$$

We investigate spherical functions on X , i.e., common eigenfunctions on X with respect to the action of the Hecke algebra $\mathcal{H}(G, K)$, where $K = G \cap GL_{2n}(\mathcal{O}_{k'})$. The Borel group B of G consisting of all the upper triangular matrices in G satisfies the Iwasawa decomposition $G = KB = BK$. The space X can be considered as the set of k -rational points of a \mathbb{G} -homogeneous \mathbb{X} , where \mathbb{G} is a reductive group defined over k such that $\mathbb{G}(k) = G$ (cf. Appendix A). In the following, we fix a prime element π in k and the absolute value $||$ on k normalized by $|\pi|^{-1} = q = \sharp(\mathcal{O}_k/(\pi))$. By a general theory, typical spherical function on X can be constructed by Poisson transform from certain relative B -invariants on X , and we introduce a spherical

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function $\omega(x; s)$ for $x \in X$ and $s \in \mathbb{C}^n$ as follows (for details, see (2.3)):

$$\omega(x; s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i + \varepsilon_i} dk, \quad (0.1)$$

where $d_i(y)$ is the determinant of the lower right i by i block of y , $\varepsilon \in \mathbb{C}^n$ is a certain fixed number, dk is the Haar measure on K . The above integral is absolutely convergent if $\operatorname{Re}(s_i) \geq \operatorname{Re}(\varepsilon_i)$, $1 \leq i \leq n$, continued to a rational function of q^{s_1}, \dots, q^{s_n} , and becomes a K -invariant function on X , hence $\omega(x; s) \in \mathcal{C}^\infty(K \backslash X)$ for each $s \in \mathbb{C}^n$.

In §1 we consider the Cartan decomposition, the K -orbit decomposition, of X . Set

$$x_\lambda = \operatorname{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}) \in X, \quad (0.2)$$

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}, \quad (0.3)$$

For $\lambda, \mu \in \Lambda_n^+$, we see

$$K \cdot x_\lambda \cap K \cdot x_\mu = \emptyset, \quad \text{if } \lambda \neq \mu,$$

since each x_λ belongs to a different $GL_{2n}(\mathcal{O}_{k'})$ -double coset in $GL_{2n}(k')$. We will show the following (cf. Theorem 1.1, Proposition 1.2, Theorem 1.8, Theorem 1.9).

Theorem 1 (1) If k has odd residual characteristic, the K -orbit decomposition of X is given by

$$X = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda. \quad (0.4)$$

(2) If k has even residual characteristic, there exists a K -orbit which does not contain any diagonal matrix, for each $n \geq 1$.

(3) There are precisely two G -orbits in X , independent of the residual characteristic of k .

In and after §2, we assume that k has odd residual characteristic, i.e. q is odd. The Weyl group W of G with respect to the maximal k -split torus in B acts on rational characters on B so does on s , and we consider the functional equations of $\omega(x; s)$ with respect to W . It is convenient to introduce a new variable $z \in \mathbb{C}^n$ which is related to s by

$$s_i = -z_i + z_{i+1}, \quad (1 \leq i \leq n), \quad s_n = -z_n, \quad (0.5)$$

and we write $\omega(x; z) = \omega(x; s)$. The Weyl group W is generated by S_n which acts on $\{z_i \mid 1 \leq i \leq n\}$ by permutation of indices and τ which acts as $\tau(z_n) = -z_n$ and $\tau(z_i) = z_i$ for $i < n$. As for z -variable, the $\mathcal{H}(G, K)$ -action on $\omega(x; z)$ can be written as (cf. (2.9))

$$(f * \omega(\cdot; z))(z) = \lambda_z(f) \omega(x; z), \quad f \in \mathcal{H}(G, K), \quad (0.6)$$

where $*$ is the convolution action of $\mathcal{H}(G, K)$ on $\mathcal{C}^\infty(K \backslash X)$, and λ_z is the Satake transform $\mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W$.

Denote by Σ_s^+ (resp. Σ_ℓ^+) the set of all positive short (resp. long) roots of G (cf. (2.19)). As for functional equations and the location of zeros and possible poles of $\omega(x; z)$, we will show the following (Theorem 2.6, Theorem 2.7):

Theorem 2 (1) For every $\sigma \in W$, one has

$$\omega(x; z) = \Gamma_\sigma(z) \cdot \omega(x; \sigma(z)),$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma_s^+(\sigma)} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}},$$

and $\Sigma_s^+(\sigma) = \Sigma_s^+ \cap \sigma(-\Sigma_s^+)$.

(2) *The function*

$$\prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle - 1}} \cdot \omega(x; z)$$

is contained in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W (= \mathcal{R}, \text{ say})$.

In §3, we will give the explicit formula for $\omega(x_\lambda; z)$ for each $\lambda \in \Lambda_n^+$ (Theorem 3.1) by a method introduced in [Hir10], which is based on functional equations of $\omega(x; z)$ and some data depending only on the group G . Since $\omega(x; z)$ is K -invariant for x , it is enough to consider the explicit formula for x_λ , $\lambda \in \Lambda_n^+$ by Theorem 1.

Theorem 3 *For each $\lambda \in \Lambda_n^+$, one has*

$$\omega(x_\lambda; z) = \frac{(1 - q^{-2})^n}{w_{2n}(-q^{-1})} \cdot \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}} \cdot c_\lambda \cdot Q_\lambda(z), \quad (0.7)$$

where

$$w_m(t) = \prod_{i=1}^m (1 - t^i), \quad c_\lambda = (-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})},$$

$$Q_\lambda(z) = \sum_{\sigma \in W} \sigma \left(q^{-\langle \lambda, z \rangle} c(z) \right), \quad c(z) = \prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}} \prod_{\alpha \in \Sigma_\ell^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}.$$

By Theorem 2, we see $Q_\lambda(z)$ is a polynomial in \mathcal{R} . On the other hand, this is a specialization of Macdonald polynomials, and it is known that the set $\{Q_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms a \mathbb{C} -basis for \mathcal{R} and $Q_0(z)$ is a constant (for details, see Appendix B). Hence, for $x_0 = 1_{2n}$, we have

$$\omega(1_{2n}; z) = \frac{(1 - q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \times \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}}. \quad (0.8)$$

In §4, we consider the spherical Fourier transform on the Schwartz space $\mathcal{S}(K \backslash X)$:

$$\begin{aligned} F: \mathcal{S}(K \backslash X) &\longrightarrow \mathcal{R} \\ \varphi &\longmapsto F(\varphi) = \int_X \varphi(x) \Psi(x; z) dx, \end{aligned}$$

where $\Psi(x; z) = \omega(x; z) / \omega(1_{2n}; z)$ and dx is a G -invariant measure on X . We will show the following (cf. Theorem 4.1, Corollary 4.2, Theorem 4.5):

Theorem 4 (1) The spherical Fourier transform F is an $\mathcal{H}(G, K)$ -module isomorphism, in particular, $\mathcal{S}(K \backslash X)$ is a free $\mathcal{H}(G, K)$ -module of rank 2^n .

(2) For each $z \in \mathbb{C}^n$, the set $\left\{ \Psi(x; z + u) \mid u \in \{0, \frac{\pi\sqrt{-1}}{\log q}\}^n \right\}$ forms a basis for the spherical functions on X corresponding to λ_z .

(3) (Plancherel formula) We give explicitly the normalization of dx on X and a measure $d\mu(z)$ on

$$\mathfrak{a}^* = \left\{ \sqrt{-1} \left(\mathbb{R} / \frac{2\pi}{\log q} \mathbb{Z} \right) \right\}^n,$$

for which

$$\int_X \varphi(x) \overline{\psi(x)} dx = \int_{\mathfrak{a}^*} F(\varphi)(z) \overline{F(\psi)(z)} d\mu(z), \quad (\varphi, \psi \in \mathcal{S}(K \backslash X)).$$

In [Hir11], we have investigated spherical functions on a similar space X_T associated with each nondegenerate hermitian matrix T , and obtained functional equations of hermitian Siegel series as an application. Both spaces, X_T and the present X , are isomorphic to $U(2n)/U(n) \times U(n)$ over the algebraic closure of k , and the former realization was useful for the application to hermitian Siegel series. But it was not easily understandable, and we could not obtain its Cartan decomposition, nor complete parametrization of spherical functions. We discuss the correspondence between both spaces in Appendix C, and see many results on the present space X are inherited to the former spaces X_T .

Throughout of this article except Appendix A, where we explain about unitary hermitian matrices in a general setting, we denote by k a non-archimedean local field of characteristic 0, fix an unramified quadratic extension k' and consider unitary and hermitian matrices with respect to k'/k . We fix a prime element π of k , denote by $v_\pi(\cdot)$ the additive value on k , and normalize the absolute value $|\cdot|$ on k^\times by $|\pi|^{-1} = q = \#(\mathcal{O}_k/(\pi))$. We also fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$. We may take ϵ such as $\epsilon - 1 \in 4\mathcal{O}_k^\times$, so that $\{1, \frac{1+\sqrt{\epsilon}}{2}\}$ forms an \mathcal{O}_k -basis for $\mathcal{O}_{k'}$ (cf. [Ome73, 64.3 and 64.4]). From §2 to §4, we assume that q is odd.

1. The space X and its K -orbit decomposition and G -orbit decomposition

Let k' be an unramified quadratic extension of a p -adic field k and consider hermitian matrices and unitary matrices with respect to k'/k . For a matrix $A \in M_{mn}(k')$, we denote by $A^* \in M_{nm}(k')$ its conjugate transpose with respect to k'/k , and say A is *hermitian* if $A^* = A$.

We consider the unitary group

$$G = G_n = \left\{ g \in GL_{2n}(k') \mid g^* j_{2n} g = j_{2n} \right\}, \quad j_{2n} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in M_{2n}, \quad (1.1)$$

the space X of unitary hermitian matrices in G

$$X = X_n = \left\{ x \in G \mid x^* = x, \Phi_{xj_{2n}}(t) = (t^2 - 1)^n \right\}, \quad (1.2)$$

and a supplementary space \tilde{X} containing X

$$\tilde{X} = \tilde{X}_n = \{x \in G \mid x = x^*\}, \quad (1.3)$$

where $\Phi_y(t)$ is the characteristic polynomial of the matrix y . The group G acts on X and \tilde{X} by

$$g \cdot x = gxg^* = x[g^*] = gxj_{2n}g^{-1}j_{2n}, \quad g \in G, x \in \tilde{X}.$$

As is explained in Appendix A, we may understand X as the set of k -rational points of a $G(\bar{k})$ -homogeneous algebraic set $X(\bar{k})$ defined over k , where \bar{k} is the algebraic closure of k .

We fix the maximal compact subgroup K of G by

$$K = K_n = G \cap M_{2n}(\mathcal{O}_{k'}). \quad (1.4)$$

The main purpose of this section is to give the Cartan decomposition of X , i.e., the K -orbit decomposition of X for odd q (Theorem 1.1), and G -orbit decomposition of X (Theorem 1.9).

To start with, we recall the case of unramified hermitian matrices. The group $G_0 = GL_n(k')$ acts on the space $\mathcal{H}_n(k') = \{y \in G_0 \mid y^* = y\}$ by $g \cdot y = gyg^*$, and there are two G_0 -orbits in $\mathcal{H}_n(k')$ determined by the parity of $v_\pi(\det(y))$. Setting $K_0 = GL_n(\mathcal{O}_{k'})$, the Cartan decomposition is known (cf. [Jac62]) as follows:

$$\mathcal{H}_n(k') = \bigsqcup_{\lambda \in \Lambda_n} K_0 \cdot \pi^\lambda, \quad (1.5)$$

where

$$\pi^\lambda = \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}), \quad \Lambda_n = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}.$$

THEOREM 1.1. *Assume that k has odd residual characteristic. Then, the K -orbit decomposition of X_n is given as follows:*

$$X_n = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda, \quad (1.6)$$

where

$$\begin{aligned} \Lambda_n^+ &= \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}, \\ x_\lambda &= \begin{pmatrix} \pi^\lambda & 0 \\ 0 & \pi^{-\lambda} \end{pmatrix} = \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}). \end{aligned}$$

For $a = (a_{ij}) \in M_{2n}(\mathcal{O}_{k'})$, we set

$$-\ell(a) = \min \{v_\pi(a_{ij}) \mid 1 \leq i, j \leq n\}, \quad (1.7)$$

and say an entry of a to be *minimal* if its v_π -value is $-\ell(a)$. For $g \in G$, we see $\ell(g) \geq 0$, since $v_\pi(\det(g)) = 0$. For regularization of $x \in \tilde{X}$, we often use elements in K of the following type:

$$\begin{aligned} &\begin{pmatrix} h & 0 \\ 0 & jh^{*-1}j \end{pmatrix}, \quad \text{for } h \in K_0, \\ &\begin{pmatrix} 1_n & aj \\ 0 & 1_n \end{pmatrix}, \quad \text{for } a \in M_n(\mathcal{O}_{k'}), a + a^* = 0. \end{aligned} \quad (1.8)$$

PROPOSITION 1.2. *Let $n = 1$. Then*

$$X_1 = \bigsqcup_{\ell \geq 0} K_1 \cdot \begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \sqcup \bigsqcup_{1 \leq r \leq v_\pi(2)} \begin{pmatrix} \pi^{-r}(1-\epsilon) & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & \pi^r \end{pmatrix},$$

where the latter union is empty if q is odd.

Proof. For $x = \begin{pmatrix} a & \beta \\ \beta^* & c \end{pmatrix} \in \tilde{X}_1$, we have

$$j_2 = j_2[x] = \begin{pmatrix} a(\beta + \beta^*) & ac + \beta^2 \\ ac + \beta^{*2} & c(\beta + \beta^*) \end{pmatrix}.$$

If $a = 0$ or $c = 0$, we have $\beta = \pm 1$, hence $a = c = 0$ and $x = \pm j_2 \notin X_1$. Hence $ac \neq 0$ for $x \in X_1$, and we may assume

$$x = \begin{pmatrix} a & b\sqrt{\epsilon} \\ -b\sqrt{\epsilon} & c \end{pmatrix}, \quad \begin{matrix} a, b, c \in k, \quad ac + b^2\epsilon = 1, \\ v_\pi(a) \geq v_\pi(c), \quad c \text{ is a power of } \pi. \end{matrix}$$

If $v_\pi(c) = -\ell \leq 0$, then $v_\pi(b) \geq -\ell$, and

$$K \cdot x \ni \begin{pmatrix} 1 & -b\pi^\ell\sqrt{\epsilon} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b\sqrt{\epsilon} \\ -b\sqrt{\epsilon} & \pi^{-\ell} \end{pmatrix} = \begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \in X_1,$$

where each ℓ gives different K_1 -orbit.

Next assume $v_\pi(c) = r > 0$, then $b \in \mathcal{O}_k^\times$, $b^2\epsilon \equiv 1 \pmod{(\pi^2)}$ and $x \in K$. Thus q is even, $b \equiv 1 \pmod{(\pi)}$ and $x \equiv j_2 \pmod{(\pi)}$, since $\epsilon \in 1 + 4\mathcal{O}_k^\times$. We may rewrite

$$x = \begin{pmatrix} \pi^r a' & 1 + \pi^m \gamma \\ 1 + \pi^m \gamma^* & \pi^r \end{pmatrix}, \quad \begin{matrix} a' \in \mathcal{O}_k, \quad 0 \leq m \leq r, \\ \gamma \in \mathcal{O}_{k'}, \quad \gamma \in \mathcal{O}_{k'}^\times \text{ if } m < r. \end{matrix}$$

Since $\Phi_{xj_2}(t) = t^2 - 1$, we have

$$\pi^m(\gamma + \gamma^*) + 2 = 0, \quad \pi^{2m}N(\gamma) = \pi^{2r}a'.$$

By the latter equation we have $m = r$, and setting $\gamma = b_0 + b_1 \frac{1+\sqrt{\epsilon}}{2}$, we have

$$1 + \pi^r \gamma = -(1 + \pi^r b_0)\sqrt{\epsilon}.$$

Thus, we obtain

$$K \cdot x \ni \begin{pmatrix} 1 & b_0\sqrt{\epsilon} \\ 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} \pi^{-r}(1 - \epsilon) & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & \pi^r \end{pmatrix} (= x_r, \text{ say}), \quad 1 \leq r \leq v_\pi(2).$$

If $K \cdot x_r$ contained a diagonal matrix it must be 1_2 , and $kx_r = j_2 k j_2$ for some $k \in K$. By the latter equation we get $\det(k) \equiv 0 \pmod{(\pi)}$, which is impossible for $k \in K$. Similarly we may prove $x_r \notin K \cdot x_s$ if $r \neq s$, which completes the proof. \square

LEMMA 1.3. *Let $n \geq 2$ and assume that $x \in \tilde{X}_n$ has a minimal entry in the diagonal. Then $K \cdot x$ contains a hermitian matrix of type*

$$\left(\begin{array}{c|c|c} \pi^\ell & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & \pi^{-\ell} \end{array} \right), \quad y \in \tilde{X}_{n-1} \cap M_{2n-2}(\pi^{-\ell}\mathcal{O}_{k'}),$$

where $\ell = \ell(x)$. If $x \in X_n$, then the above $y \in X_{n-1}$.

Proof. By the action of W , we may assume the $(2n, 2n)$ -entry is minimal. Then, by (1.5) and (1.8), we see there is some $x' \in K \cdot x$ whose lower right n by n block has the form

$$\left(\begin{array}{c|c} * & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & \pi^{-\ell} \end{array} \right).$$

Then, by taking a suitable matrix of type

$$h = \begin{pmatrix} 1_n & A \\ 0 & 1_n \end{pmatrix} \in K, \quad A = \left(\begin{array}{c|c} -a_n^* \cdots -a_2^* & a \\ \hline 0 & \begin{matrix} a_2 \\ \vdots \\ a_n \end{matrix} \end{array} \right) \in M_n(k'), \quad a + a^* = 0,$$

$h \cdot x'$ becomes the following form

$$\left(\begin{array}{c|ccc|c} c & c_2 & \cdots & c_{2n-1} & b \\ \hline c_2^* & & & & 0 \\ \vdots & & y & & \vdots \\ c_{2n-1}^* & & & & 0 \\ \hline b^* & 0 & \cdots & 0 & \pi^{-\ell} \end{array} \right) \in \tilde{X}_n, \quad \begin{aligned} b &= \pi^{-\ell}(b_0 + b_1 \frac{1+\sqrt{\epsilon}}{2}), \text{ with} \\ b_0, b_1 &\in \mathcal{O}_k, \quad b_1 = 0 \text{ or } b_1 \notin 2\mathcal{O}_k, \\ c &\in \pi^{-\ell}\mathcal{O}_k, \quad c_i \in \pi^{-\ell}\mathcal{O}_{k'}. \end{aligned} \quad (1.9)$$

Since $j_{2n}[h \cdot x'] = j_{2n}$, we have

$$b = 0, \quad c = \pi^\ell, \quad c_i = 0 \quad (2 \leq i \leq 2n-1),$$

and then $y \in \tilde{X}_{n-1}$, and it is clear that $y \in X_{n-1}$ if $x \in X_n$, which completes the proof. \square

LEMMA 1.4. *Let $n \geq 2$ and assume that $x \in \tilde{X}_n$ has a minimal entry outside of the diagonal and the anti-diagonal. Then $K \cdot x$ contains a hermitian matrix of type*

$$\left(\begin{array}{cc|c|c} \pi^\ell & 0 & 0 & 0 \\ 0 & \pi^\ell & 0 & 0 \\ \hline 0 & y & 0 & 0 \\ \hline 0 & 0 & \pi^{-\ell} & 0 \\ & & 0 & \pi^{-\ell} \end{array} \right), \quad y \in \tilde{X}_{n-2} \cap M_{2n-4}(\pi^{-\ell}\mathcal{O}_{k'}),$$

where $\ell = \ell(x)$. If $x \in X_n$, the the above $y \in X_{n-2}$.

Proof. By the assumption, the minimal entries appear in pair not in the anti-diagonal. Then, by the action of W , we may assume the $(2n, 2n-1)$ -entry and the $(2n-1, 2n)$ -entry are minimal. Then, by (1.5) and (1.8), we see there is some $x' \in K \cdot x$ whose lower right n by n block has the form

$$\left(\begin{array}{c|cc} & 0 & 0 \\ & \vdots & \vdots \\ * & 0 & 0 \\ \hline 0 \cdots 0 & \pi^{-\ell} & 0 \\ 0 \cdots 0 & 0 & \pi^{-\ell} \end{array} \right).$$

Taking the similar procedure (twice) to the proof of Lemma 1.3, we obtain a matrix of the required form. \square

LEMMA 1.5. *Let $x \in \tilde{X}_n$ with $n \geq 2$. Assume that any minimal entry of $x \in \tilde{X}_n$ stands in the anti-diagonal but not all the entries of the anti-diagonal are minimal. Then $K \cdot x$ contains a hermitian matrix of the same type as in Lemma 1.4.*

Proof. Set $\ell = \ell(x)$. By the action of W , we may assume

$$x = \left(\begin{array}{c|cc} & a & \xi \\ * & c & b \\ \hline * & & * \end{array} \right), \quad v_\pi(\xi) = -\ell, \quad a, b, c \in \pi^{-\ell+1}\mathcal{O}_{k'}.$$

Then, for the matrix

$$h = \left(\begin{array}{cc|c|cc} 1 & 1 & & & \\ 0 & 1 & & & \\ \hline & & 0 & & \\ & & 1_{2n-4} & & \\ \hline & & & 1 & -1 \\ & & & 0 & 1 \end{array} \right) \in K,$$

$\ell(h \cdot x) = \ell(x)$ and the $(1, 2n-1)$ -entry of $h \cdot x$ is equal to $-\xi + a - b + c$ and minimal. Then, $h \cdot x$ satisfies the assumption of Lemma 1.4, and the result follows from this. \square

LEMMA 1.6. *Let $x \in \tilde{X}_n$ with $n \geq 2$. Assume that any minimal entry of x stands in the anti-diagonal and that all the entries in the anti-diagonal are minimal. Denote by ξ_i the $(i, 2n-i)$ -entry of x , ($1 \leq i \leq n$). Then*

(i) *On has $\ell(x) = 0$, $x \in K$, and $\xi_i \equiv \pm 1 \pmod{(\pi)}$, $1 \leq i \leq n$.*

(ii) *If $\xi_i \not\equiv \xi_j \pmod{(\pi)}$ for some i and j , which occurs only when $2 \notin (\pi)$, then $K \cdot x$ contains a hermitian matrix of type*

$$\left(\begin{array}{c|c|c} 1_2 & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & 1_2 \end{array} \right) \in K, \quad y \in \tilde{X}_{n-2} \cap M_{2n-4}(\mathcal{O}_{k'}),$$

where we understand the above matrix is 1_4 when $n = 2$. If $X \in X_n$, then the above $y \in X_{n-2}$.

(iii) *If k has odd residual characteristic and $x \equiv \pm j_{2n} \pmod{(\pi)}$, then $x \notin X_n$.*

Proof. (i) By the assumption, we see $v_\pi(\det x) = 2v_\pi(\xi_1 \cdots \xi_n) = -2\ell n$, which must be 0, hence $\ell = 0$ and

$$x \equiv \begin{pmatrix} & & & \xi_1 \\ & 0 & & \ddots \\ & & \xi_n & \\ & & \xi_n^* & \\ \ddots & & & 0 \\ \xi_1^* & & & \end{pmatrix} \pmod{(\pi)}.$$

Since $j_{2n}[x] = j_{2n}$, we have $\xi_i^2 \equiv \xi_i^{*2} \equiv 1 \pmod{(\pi)}$, hence $\xi_i \equiv \pm 1 \pmod{(\pi)}$ for every i .

(ii) Now we assume $2 \notin (\pi)$ and $\xi_i \not\equiv \xi_j \pmod{(\pi)}$ for some i and j . We may assume $\xi_1 \not\equiv \xi_2 \pmod{(\pi)}$ by the action of W , and write

$$x = \left(\begin{array}{cc|c|cc} a & b & & c & \xi_1 \\ b^* & d & * & \xi_2 & f \\ \hline & & * & & \\ \hline c^* & \xi_2^* & & g & h \\ \xi_1^* & f^* & * & h^* & r \end{array} \right).$$

For $h \in K$ with

$$h = \left(\begin{array}{cc|c|cc} 1 & 0 & & & \\ 0 & 1 & & & \\ \hline & & 0 & & \\ & & 1_{2n-4} & & \\ \hline 1 & 0 & & 1 & 0 \\ 0 & -1 & & 0 & 1 \end{array} \right) \in K,$$

the lower right 2 by 2 block of $h \cdot x$ becomes

$$\begin{pmatrix} a + c + c^* + g & -b + h + \xi_1 - \xi_2^* \\ -b^* + h^* + \xi_1^* - \xi_2 & d - f - f^* + r \end{pmatrix} \equiv \begin{pmatrix} 0 & 2\xi_1 \\ 2\xi_1 & 0 \end{pmatrix} \pmod{(\pi)},$$

which is unimodular hermitian of size 2, since $2 \notin (\pi)$, and we may change it into 1_2 . Then, the similar procedure (twice) to the proof of Lemma 1.3, we see $K \cdot x$ contains an element of the required form.

(iii) If q is odd and $x \equiv cj_n \pmod{(\pi)}$ with $c = \pm 1$, then $\Phi_{xj_{2n}}(t) \equiv (t - c)^{2n} \not\equiv (t^2 - 1)^n \pmod{(\pi)}$, hence $x \notin X_n$. \square

Now Theorem 1.1 follows from Lemmas 1.3 to 1.6.

We consider even residual case.

LEMMA 1.7. *Assume that q is even and $x \in X_n \cap K$ satisfies $x \equiv j_{2n} \pmod{(\pi)}$. Then, $K \cdot x$ does not contain any diagonal matrix, and represented by a matrix of the following type:*

$$E_n(\mu) = \begin{pmatrix} \pi^{-\mu_n}(1 - \epsilon) & & & & -\sqrt{\epsilon} \\ & \ddots & & & \\ & & \pi^{-\mu_1}(1 - \epsilon) & -\sqrt{\epsilon} & \\ & & \sqrt{\epsilon} & \pi^{\mu_1} & \\ & & & & \ddots \\ \sqrt{\epsilon} & & & & & \pi^{\mu_n} \end{pmatrix},$$

where each empty place means zero-entry, and

$$\mu \in \Lambda_n^{(2)} = \{ \mu \in \Lambda_n^+ \mid v_\pi(2) \geq \mu_1 \geq \dots \geq \mu_n \geq 1 \}.$$

Proof. If $K \cdot x$ contains a diagonal matrix, it must contain 1_{2n} , since $x \in K$. Assume $k \cdot x = 1_{2n}$ for some $k \in K$, and write

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in M_n(\mathcal{O}_{k'}).$$

Then, since k satisfies $kx = j_{2n}kj_{2n}$, we have

$$k \equiv \begin{pmatrix} a & b \\ j_na & j_nb \end{pmatrix} \pmod{(\pi)},$$

which contradicts to the fact $\det(k) \in \mathcal{O}_{k'}^\times$, hence $K \cdot x$ does not contain any diagonal matrix.

Next we will show that $K \cdot x$ contains an $E_n(\mu)$ of the above type. Since $j_{2n} \notin X_n$, we may write

$$x = j_{2n} + \pi^m y, \quad m > 0, \quad 0 \neq y = y^* \in M_{2n}(\mathcal{O}_{k'}), \quad \ell(y) = 0, \quad (1.10)$$

where $\ell(y) = 0$ means that minimal entry of y is a unit. If any entry in the anti-diagonal of y is a unit and all the other entries contained in (π) , then $\det(y) \in \mathcal{O}_k^\times$. On the other hand

$$(t^2 - 1)^n = \Phi_{xj_{2n}}(t) = \det(t1_{2n} - (j_{2n} + \pi^m y)j_{2n}) = \det((t - 1)1_{2m} - \pi^m yj_{2n}),$$

and

$$0 = \Phi_{xj_{2n}}(1) = \det(-\pi^m yj_{2n}) = (-1)^n \pi^{2mn} \det(y),$$

which is a contradiction. Hence there is a minimal entry of y not in the anti-diagonal. Then following the proof of Lemmas 1.3 to 1.5, there exists some $k \in K$, for which $k \cdot y$ becomes (1.9) with $\ell = 0$. Since $k \cdot x = j_{2n} + \pi^m(k \cdot y)$ satisfies $(k \cdot x) \cdot j = j$, looking at the $2n$ -th row, we have

$$c_i = 0, i \geq 2, \quad 2 + 2\pi^m b_0 + \pi^m b_1 = 0,$$

and

$$k \cdot x = \left(\begin{array}{c|c|c} \pi^m c & 0 & -(1 + \pi^m b_0)\sqrt{\epsilon} \\ \hline 0 & \xi & 0 \\ \hline (1 + \pi^m b_0) & 0 & \pi^m \end{array} \right), \quad \xi \equiv j_{2(n-1)} \pmod{(\pi^m)},$$

Then, by acting

$$\left(\begin{array}{c|c|c} 1 & 0 & b_0\sqrt{\epsilon} \\ \hline 0 & 1_{2(n-1)} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \in K,$$

$k \cdot x$ becomes

$$\left(\begin{array}{c|c|c} \pi^m(1 - \epsilon) & 0 & -\sqrt{\epsilon} \\ \hline 0 & \xi & 0 \\ \hline \sqrt{\epsilon} & 0 & \pi^m \end{array} \right), \quad \xi \equiv j_{2(n-1)} \pmod{(\pi^m)}.$$

Repeating the same procedure, we conclude the proof. \square

Now, by Lemmas 1.3 to 1.5, Lemma 1.6-(i), and Lemma 1.7, we see the following.

THEOREM 1.8. *Assume that q is even. Then*

$$X_n = \bigcup_{r=0}^n \bigcup_{\substack{\lambda \in \Lambda_r^+ \\ \mu \in \Lambda_{n-r}^{(2)}}} K \cdot x_{\lambda, \mu}, \quad x_{\lambda, \mu} = \begin{pmatrix} D_r(\lambda) & 0 & 0 \\ 0 & E_{n-r}(\mu) & 0 \\ 0 & 0 & D_r(-\lambda) \end{pmatrix},$$

where $D_r(\lambda)$ and $D_r(-\lambda)$ are related to x_λ by

$$x_\lambda = \begin{pmatrix} D_r(\lambda) & 0 \\ 0 & D_r(-\lambda) \end{pmatrix} \in X_r,$$

and $x_{\lambda, \mu}$ is understood as x_λ (resp. $E_n(\mu)$) if $r = n$ (resp. $r = 0$). Further one has

$$\bigcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda \not\supset E_n(\mu) \quad (\mu \in \Lambda_n^{(2)}).$$

As a corollary of Theorem 1.1 and Theorem 1.8, we have the following. For $\lambda \in \Lambda_n^+$, we set $|\lambda| = \sum_{i=1}^n \lambda_i$ and call λ to be *even* or *odd* according to the parity of $|\lambda|$.

THEOREM 1.9. *There are precisely two G -orbits in X_n :*

$$X_n = G \cdot x_0 \sqcup G \cdot x_1, \quad x_0 = 1_{2n}, \quad x_1 = \text{Diag}(\pi, 1, \dots, 1, \pi^{-1}). \quad (1.11)$$

If q is odd, then

$$G \cdot x_0 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ \text{even}}} K \cdot x_\lambda, \quad G \cdot x_1 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ \text{odd}}} K \cdot x_\lambda.$$

If q is even, $x_{\lambda, \mu}$ is G -equivalent to x_0 if and only if $|\lambda| + |\mu|$ is even.

Proof. For unramified hermitian matrices, it is known that $\mathcal{H}_m(k')$ has two $GL_m(k')$ -orbits determined by the parity of $v_\pi(\det(y))$ and $H^1(\Gamma, U(y)(\bar{k})) \cong C_2$, where $\Gamma = \text{Gal}(\bar{k}/k)$, $y \in \mathcal{H}_m(k')$ and $m \geq 1$. If q is odd, each representative x_λ of K -orbit in Theorem 1.1 is diagonal; if q is even, by the action of B , $x_{\lambda,\mu}$ in Theorem 1.8 becomes diagonal, hence there are at most two G -orbits in X_n , independent of the parity of q . We recall $G(\bar{k})$, $X_n(\bar{k})$ and \star -action in (A.3) in Appendix A for $m = 2n$, and set

$$H(\bar{k}) = \{ h \in G(\bar{k}) \mid h \star 1_{2n} = 1_{2n} \},$$

then it is easy to see

$$\begin{aligned} H(\bar{k}) &= \left\{ \begin{pmatrix} a & b \\ jbj & jaj \end{pmatrix} \in GL_{2n}(\bar{k}) \mid a + bj, a - bj \in U(1_n)(\bar{k}) \right\} \quad (j = j_n) \\ &\cong U(1_n)(\bar{k}) \times U(1_n)(\bar{k}). \end{aligned} \quad (1.12)$$

By the exact sequence of Γ -sets

$$\begin{array}{ccccccc} 1 & \longrightarrow & H(\bar{k}) & \longrightarrow & G(\bar{k}) & \longrightarrow & X_n(\bar{k}) \longrightarrow 1, \\ & & & & g & \longmapsto & g \star 1_{2n} \end{array}$$

we have an exact sequence of pointed sets (cf. [Ser64, I-§5.4])

$$1 \longrightarrow G \cdot 1_{2n} \longrightarrow X_n \longrightarrow H^1(\Gamma, H(\bar{k})) \xrightarrow{\eta} H^1(\Gamma, G(\bar{k})).$$

Since η is a map from $C_2 \times C_2$ to C_2 , $\text{Ker}(\eta)$ cannot be trivial and $G \cdot 1_{2n} \neq X_n$. Hence there are at least two G -orbits in X_n , thus exactly two G -orbits and they are given as above. \square

2. Spherical function $\omega(x; s)$ on X

2.1. For simplicity, we write $j = j_n$, and take a Borel subgroup B of G by

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & j b^{*-1} j \end{pmatrix} \begin{pmatrix} 1_n & aj \\ 0 & 1_n \end{pmatrix} \in G \mid \begin{array}{l} b \text{ is upper triangular of size } n \\ a + a^* = 0 \end{array} \right\},$$

where B consists of all the upper triangular matrices in G .

We introduce a spherical function $\omega(x; s)$ on X by Poisson transform from relative B -invariants. For a matrix $g \in G$, denote by $d_i(g)$ the determinant of lower right i by i block of g . Then $d_i(x)$, $1 \leq i \leq n$ are relative B -invariants on X associated with rational characters ψ_i of B , where

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)), \quad (x \in X, p \in B). \quad (2.1)$$

We set

$$X^{op} = \{ x \in X \mid d_i(x) \neq 0, 1 \leq i \leq n \}. \quad (2.2)$$

For $x \in X$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x; s) = \int_K |\mathbf{d}(k \cdot x)|^{s+\varepsilon} dk, \quad |\mathbf{d}(y)|^s = \prod_{i=1}^n |d_i(y)|^{s_i}, \quad (2.3)$$

where dk is the normalized Haar measure on K , k runs over the set $\{k \in K \mid k \cdot x \in X^{op}\}$, and

$$\varepsilon = \varepsilon_0 + \left(\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q}\right), \quad \varepsilon_0 = (-1, \dots, -1, -\frac{1}{2}) \in \mathbb{C}^n. \quad (2.4)$$

The right hand side of (2.3) is absolutely convergent if $\operatorname{Re}(s_i) \geq -\operatorname{Re}(\varepsilon_i) = -\varepsilon_{0,i}$, $1 \leq i \leq n$, and continued to a rational function of q^{s_1}, \dots, q^{s_n} , and we use the notation $\omega(x; s)$ in such sense. We note here that

$$|\psi(p)|^\varepsilon = \left(\prod_{i=1}^n |\psi_i(p)|^{\varepsilon_i}\right) = |\psi(p)|^{\varepsilon_0} = \delta^{\frac{1}{2}}(p), \quad (2.5)$$

where δ is the modulus character on B (i.e., $d(pp') = \delta(p')^{-1}dp$ for the left invariant measure dp on B).

By a general theory, the function $\omega(x; s)$ becomes an $\mathcal{H}(G, K)$ -common eigenfunction on X (cf. [Hir99, §1], or [Hir10, §1]), and we call it a spherical function on X . More precisely, the Hecke algebra $\mathcal{H}(G, K)$ of G with respect to K is the commutative \mathbb{C} -algebra consisting of compactly supported two-sided K -invariant functions on G , which acts on the space $\mathcal{C}^\infty(K \backslash X)$ of left K -invariant functions on X by

$$(f * \Psi)(y) = \int_G f(g) \Psi(g^{-1} \cdot y) dg, \quad (f \in \mathcal{H}(G, K), \Psi \in \mathcal{C}^\infty(K \backslash X)), \quad (2.6)$$

where dg is the Haar measure on G normalized by $\int_K dk = 1$, and we see

$$(f * \omega(\cdot; s))(x) = \lambda_s(f) \omega(x; s), \quad (f \in \mathcal{H}(G, K)), \quad (2.7)$$

where λ_s is the \mathbb{C} -algebra homomorphism defined by

$$\begin{aligned} \lambda_s : \mathcal{H}(G, K) &\longrightarrow \mathbb{C}(q^{s_1}, \dots, q^{s_n}), \\ f &\longmapsto \int_B f(p) |\psi(p)|^{-s+\varepsilon} dp. \end{aligned}$$

We introduce a new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n \quad (2.8)$$

and write $\omega(x; z) = \omega(x; s)$. Denote by W the Weyl group of G with respect to the maximal k -split torus in B . Then W acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1} b n_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega(x; s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices, and W is generated by S_n and $\tau : (z_1, \dots, z_n) \longmapsto (z_1, \dots, z_{n-1}, -z_n)$. Keeping the relation (2.8), we also write $\lambda_z(f) = \lambda_s(f)$. Since

$$|\psi(p)|^{-s+\varepsilon} = \prod_{i=1}^n |N(p_i)|^{-z_i} \times \delta^{\frac{1}{2}}(p),$$

where p_i is the i -th diagonal component of $p \in B$, the \mathbb{C} -algebra map λ_z is an isomorphism (the Satake isomorphism)

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W, \quad (2.9)$$

where the ring of the right hand side is the invariant subring of the Laurent polynomial ring $\mathbb{C}[q^{2z_1}, q^{-2z_1}, \dots, q^{2z_n}, q^{-2z_n}]$ by W .

By using a result on spherical functions on the space of hermitian forms, we obtain the following results.

THEOREM 2.1. *The function $G_1(z) \cdot \omega(x; z)$ is invariant under the action of S_n on z , where*

$$G_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}}. \quad (2.10)$$

Proof. By the embedding

$$K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K, \quad h \longmapsto \tilde{h} = \begin{pmatrix} jh^{*-1}j & 0 \\ 0 & h \end{pmatrix},$$

and the normalized Haar measure dh on K_0 , we obtain, for $s \in \mathbb{C}^n$ satisfying $\operatorname{Re}(s_i) \geq -\operatorname{Re}(\varepsilon_i)$, $1 \leq i \leq n$,

$$\begin{aligned} \omega(x; z) &= \omega(x; s) = \int_{K_0} dh \int_K |\mathbf{d}(k \cdot x)|^{s+\varepsilon} dk \\ &= \int_{K_0} dh \int_K |\mathbf{d}(\tilde{h}k \cdot x)|^{s+\varepsilon} dk = \int_K \int_{K_0} |\mathbf{d}(\tilde{h}k \cdot x)|^{s+\varepsilon} dh dk \\ &= \int_K \zeta_*^{(h)}(D(k \cdot x); z) dk. \end{aligned} \quad (2.11)$$

Here $D(k \cdot x)$ is the lower right n by n block of $k \cdot x$ for $\{k \in K \mid k \cdot x \in X^{op}\}$, and $\zeta_*^{(n)}(y; z)$ is a spherical function on $\mathcal{H}_n(k')$ defined by

$$\zeta_*^{(h)}(y; z) = \int_{K_0} |\mathbf{d}(h \cdot y)|^{s+\varepsilon} dh, \quad (h \cdot y = hyh^*),$$

where the variable z is related to s by (2.8), ε is defined in (2.4), and h runs over the set $\{h \in K_0 \mid d_i(h \cdot y) \neq 0, 1 \leq i \leq n\}$. The assertion of Theorem 2.1 follows from the next proposition. \square

PROPOSITION 2.2. *For any $y \in \mathcal{H}_n(k')$, the function $G_1(z) \cdot \zeta_*^{(h)}(y; z)$ is holomorphic on \mathbb{C}^n and invariant under the action of S_n .*

Proof. In [Hir05, §4.2], we have considered the following spherical function on $\mathcal{H}_n(k')$

$$\zeta^{(h)}(y; z) = |\det(y)|^{\frac{n}{2}} \int_{K_0} \prod_{i=1}^n |\hat{d}_i(h \cdot y)|^{s_i + \varepsilon_i} dh,$$

where $\hat{d}_i(y)$ is the determinant of upper left i by i block of y , and the relation of z and s and ε are the same as before, and showed that the function $G_1(z) \cdot \zeta^{(h)}(y; z)$ is holomorphic on \mathbb{C}^n and invariant under the action of S_n . Since $d_i(y) = \det(y) \hat{d}_{n-i}(y^{-1})$, we see

$$\begin{aligned} \zeta_*^{(h)}(y; z) &= \int_{K_0} |\det(h \cdot y)|^{\sum_{i=1}^n (s_i + \varepsilon_i)} \prod_{i=1}^{n-1} |\hat{d}_{n-i}(h^{*-1} \cdot y^{-1})|^{s_i + \varepsilon_i} dh \\ &= |\det(y^{-1})|^{-\sum_i (s_i + \varepsilon_i)} \cdot \int_{K_0} \prod_{i=1}^{n-1} |\hat{d}_i(h \cdot y^{-1})|^{s_{n-i} + \varepsilon_i} dh \\ &= \zeta^{(h)}(y^{-1}; w), \end{aligned}$$

where w is the z -variable corresponding to the s -variable $(s_{n-1}, \dots, s_1, -(s_1 + \dots + s_n) + \frac{n}{2} + (n-1)\frac{\pi\sqrt{-1}}{\log q})$ under the relation (2.8). Then $w_i - w_j = z_{n-j+1} - z_{n-i+1}$ for $1 \leq i < j \leq n$,

and $G_1(w) = G_1(z)$. Hence $G_1(z) \cdot \zeta_*^{(h)}(y; z) = G_1(w) \cdot \zeta^{(h)}(y^{-1}; w)$ is holomorphic and S_n -invariant. \square

2.2. Hereafter till the end of §4, we assume k has odd residual characteristic, i.e. q is odd. In this subsection, we give the functional equation of $\omega(x; s)$ for $\tau \in W$.

THEOREM 2.3. *For general size n , the spherical function satisfies the functional equation*

$$\omega(x; z) = \omega(x; \tau(z)).$$

First we consider for $\omega^{(1)}(x; s)$, the case of size $n = 1$, where $z = -s$ and τ acts as $\tau(s) = -s$.

PROPOSITION 2.4. *For $x_\ell = \begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \in X_1$, $\ell \geq 0$, one has*

$$\omega^{(1)}(x_\ell; s) = \frac{(-1)^\ell q^{-\frac{\ell}{2}}}{1 + q^{-1}} \times \left(\frac{q^{\ell s}(1 - q^{-2s-1})}{1 - q^{-2s}} + \frac{q^{-\ell s}(1 - q^{2s-1})}{1 - q^{2s}} \right),$$

in particular, $\omega^{(1)}(x; s)$ is holomorphic on \mathbb{C} and satisfies the functional equation

$$\omega^{(1)}(x; s) = \omega^{(1)}(x; -s).$$

Proof. It is easy to see

$$\begin{aligned} K &= K_1 = K_{1,1} \sqcup K_{1,2}, \quad \text{where} \\ K_{1,1} &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & v/\sqrt{\epsilon} \\ u\sqrt{\epsilon} & 1 + uv \end{pmatrix} \mid \alpha \in \mathcal{O}_{k'}^\times, u, v \in \mathcal{O}_k \right\}, \\ K_{1,2} &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} \pi u \sqrt{\epsilon} & 1 + \pi uv \\ 1 & v/\sqrt{\epsilon} \end{pmatrix} \mid \alpha \in \mathcal{O}_{k'}^\times, u, v \in \mathcal{O}_k \right\}, \end{aligned} \quad (2.12)$$

and $\text{vol}(K_{1,1}) = \frac{1}{1+q^{-1}}$ and $\text{vol}(K_{1,2}) = \frac{q^{-1}}{1+q^{-1}}$ with respect to the measure dh on K normalized by $\text{vol}(K) = 1$. Let x_ℓ be as above. For $h \in K_{1,1}$ written as above, we have

$$\begin{aligned} d_1(h \cdot x_\ell) &= N(\alpha)^{-1} \pi^{-\ell} ((1 + uv)^2 - \pi^{2\ell} u^2 \epsilon), \\ v_\pi(d_1(h \cdot x_\ell)) &= \begin{cases} -\ell & \text{if } u \in (\pi), \\ -\ell + 2 \min\{v_\pi(1 + uv), \ell\} & \text{if } u \in \mathcal{O}_k^\times, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\int_{K_{1,1}} |d_1(h \cdot x_\ell)|^{s - \frac{1}{2} + \frac{\pi\sqrt{-1}}{\log q}} dh \\ &= \frac{(-1)^\ell q^{\ell(s - \frac{1}{2})}}{1 + q^{-1}} \left(q^{-1} + (1 - q^{-1}) \left((1 - q^{-1}) + \sum_{k=1}^{\ell-1} q^{-2k(s - \frac{1}{2})} q^{-k} (1 - q^{-1}) + q^{-2\ell(s - \frac{1}{2})} q^{-\ell} \right) \right) \\ &= \frac{(-1)^\ell q^{-\frac{\ell}{2}}}{1 + q^{-1}} \left(q^{\ell s - 1} + (1 - q^{-1}) q^{-\ell s} + (1 - q^{-1})^2 \frac{q^{\ell s} - q^{-\ell s}}{1 - q^{-2s}} \right). \end{aligned}$$

For $h \in K_{1,2}$ written as above, we have

$$\begin{aligned} d_1(h \cdot x_\ell) &= N(\alpha)^{-1} \pi^{-\ell} (\pi^{2\ell} - v^2/\epsilon), \\ v_\pi(d_1(h \cdot x_\ell)) &= -\ell + 2 \min\{v_\pi(v), \ell\}, \end{aligned}$$

and

$$\begin{aligned}
 & \int_{K_{1,2}} |d_1(h \cdot x_\ell)|^{s-\frac{1}{2}+\frac{\pi\sqrt{-1}}{\log q}} dh \\
 &= \frac{(-1)^\ell q^{\ell(s-\frac{1}{2})} q^{-1}}{1+q^{-1}} \left(\sum_{k=0}^{\ell-1} q^{-2k(s-\frac{1}{2})} q^{-k} (1-q^{-1}) + q^{-2\ell(s-\frac{1}{2})} q^{-\ell} \right) \\
 &= \frac{(-1)^\ell q^{-\frac{\ell}{2}}}{1+q^{-1}} \left(q^{-\ell s-1} + (q^{-1}-q^{-2}) \frac{q^{\ell s}-q^{-\ell s}}{1-q^{-2s}} \right)
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 \omega(x_\ell; s) &= \int_{K_1} |d_1(k \cdot x_\ell)|^{s-\frac{1}{2}-\frac{\pi\sqrt{-1}}{\log q}} dk \\
 &= \frac{1}{1+q^{-1}} \left\{ \sum_{r=0}^{\ell} (-1)^\ell q^{-(2r-\ell)(s-\frac{1}{2})} q^{-r} (1-q^{-1}) + (-1)^\ell q^{-\ell(s-\frac{1}{2})} q^{-(\ell+1)} \right\} \\
 &\quad + \frac{q^{-1}}{1+q^{-1}} (-1)^\ell \cdot q^{\ell(s-\frac{1}{2})} \\
 &= \frac{(-1)^\ell q^{-\frac{\ell}{2}}}{1+q^{-1}} \left(\frac{q^{\ell s}(1-q^{-2s-1})}{1-q^{-2s}} + \frac{q^{-\ell s}(1-q^{2s-1})}{1-q^{2s}} \right) \\
 &= \frac{(-1)^\ell q^{-\frac{\ell}{2}}}{1+q^{-1}} \frac{1}{q^s - q^{-s}} \left(q^{(\ell+1)s} - q^{-(\ell+1)s} - q^{-1}(q^{(\ell-1)s} - q^{-(\ell-1)s}) \right),
 \end{aligned}$$

which is holomorphic and invariant under $s \mapsto -s$. Since the set $\{x_\ell \mid \ell \geq 0\}$ forms a set of complete representatives of $K_1 \backslash X_1$ (cf. Proposition 1.2), we conclude the proof. \square

We assume $n \geq 2$. As for the functional equation for τ , we may follow a similar line as in [Hir11, §2.2]. Since $G(=U(j_{2n})) = \tilde{j}_n U(H_n) \tilde{j}_n$, where $H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ and $\tilde{j}_n = \begin{pmatrix} 1_n & 0 \\ 0 & j_n \end{pmatrix}$, we have to modify everything by this conjugation. Set

$$w_\tau = \begin{pmatrix} 1_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1_{n-1} \end{pmatrix},$$

then the standard parabolic subgroup P attached to τ is given as follows

$$\begin{aligned}
 P &= B \cup B w_\tau B \\
 &= \left\{ \left(\begin{array}{c|c} q' & b \\ \hline a & d \\ \hline c & q \end{array} \right) \left(\begin{array}{cc|c} 1_{n-1} & \alpha & 0 \\ \hline & 1 & -\alpha^* j \\ \hline 0 & 1 & 1_{n-1} \end{array} \right) \left(\begin{array}{cc|c} 1_{n-1} & \beta & \gamma j \\ \hline & 0 & -\beta^* j \\ \hline 0 & 1 & 1_{n-1} \end{array} \right) \in G \mid \right. \\
 &\quad \left. \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'), q' = jq^{*-1}j \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(j_2), \alpha, \beta \in M_{n-1,1}(k'), \\ \gamma \in M_{n-1}(k'), \gamma + \gamma^* = 0 \end{array} \right\}, \tag{2.13}
 \end{aligned}$$

where $j = j_{n-1}$ and each empty place in the above expression means zero-entry.

We consider the following action of $\tilde{P} = P \times GL_1(k')$ on $\tilde{X} = X \times V$ with $V = M_{21}(k')$:

$$(p, r) \star (x, v) = (p \cdot x, \rho(p)vr^{-1}), \quad (p, r) \in \tilde{P}, \quad (x, v) \in \tilde{X},$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the decomposition of $p \in P$ as in (2.13). We define

$$g(x, v) = \det \left[\left(\begin{array}{c|c} \begin{matrix} -v_2 & v_1 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & 1_{n-1} \end{array} \right) \cdot x_{(n+1)} \right], \quad (x, v) \in \tilde{X}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (2.14)$$

where $x_{(n+1)}$ is the lower right $(n+1)$ by $(n+1)$ block of x . Then we have the following.

LEMMA 2.5. *Let $g(x, v)$ be the function on \tilde{X} defined by (2.14).*

(i) *$g(x, v)$ is a \tilde{P} -relative invariant on \tilde{X} associated with the \tilde{P} -rational character $\tilde{\psi}(p, r) = \psi_{n-1}(p)N(r)^{-1}$, and $g(x, v_0) = d_n(x)$ with $v_0 = {}^t(1\ 0)$.*

(ii) *$g(x, v)$ is expressed as $g(x, v) = D(x)[v]$ by some hermitian matrix $D(x)$ of size 2. For $x \in X^{op}$, $D_1(x) = d_{n-1}(x)^{-1}D(x)$ belongs to X_1 .*

Proof. (i) It is easy to see that $g(x, v_0) = d_n(x)$ and $g((1, r) \star (x, v)) = N(r)^{-1}g(x, v)$. Take an element p in P and write as

$$p = \left(\begin{array}{c|c|c} q' & \alpha' & \gamma \\ \hline 0 & \rho(p) & \alpha \\ \hline 0 & 0 & q \end{array} \right), \quad (q, q', \gamma \in M_{n-1}, \alpha, {}^t\alpha' \in M_{2,n}, \rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(j_2)).$$

Then

$$\begin{aligned} g((p, 1) \star (x, v)) &= \det \left[\left(\begin{array}{c|c} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & 0 \\ \hline 0 & 1_{n-1} \end{array} \right) \cdot \left(\begin{array}{c|c} \rho(p) & \alpha \\ \hline 0 & q \end{array} \right) \cdot x_{(n+1)} \right] \\ &= \det \left[\left(\begin{array}{c|c} \begin{pmatrix} u(-v_2) & v_1 \\ 0 & \beta \end{pmatrix} & \\ \hline 0 & q \end{array} \right) \cdot x_{(n+1)} \right] \\ &= \det \left[\left(\begin{array}{c|c} \begin{pmatrix} u & \beta \\ 0 & q \end{pmatrix} & \\ \hline 0 & 1_{n-1} \end{array} \right) \cdot x_{(n+1)} \right] \\ &= N(\det(q))g(x, v) \\ &= \psi_{n-1}(p)g(x, v), \end{aligned}$$

where $u = \det(\rho(p)) \in \mathcal{O}_{k'}^1 (= \{u \in \mathcal{O}_{k'}^\times \mid N(u) = 1\})$ and $\beta = (-v_2 \ v_1) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \alpha \in M_{1n}(k')$.

Hence we see that

$$g((p, r) \star (x, v)) = \psi_{n-1}(p)N(r)^{-1}g(x, v), \quad (p, r) \in \tilde{P}. \quad (2.15)$$

(ii) Since $g(x, v)$ is a linear form with respect to in v_1, v_2 and v_1^*, v_2^* and $g(x, v)^* = g(x, v)$, it is written as $D(x)[v]$ for some hermitian matrix $D(x)$ of size 2. For $x = \text{Diag}(a_1^{-1}, \dots, a_n^{-1}, a_n, \dots, a_1) \in X^{op}$, we have

$$\begin{aligned} g(x, v) &= \det \left[\left(\begin{array}{c|c} \begin{pmatrix} -v_2 & v_1 \\ 0 & 0 \end{pmatrix} & 0 \\ \hline 0 & 1_{n-1} \end{array} \right) \cdot \text{Diag}(a_n^{-1}, a_n, \dots, a_1) \right] \\ &= (a_1 \cdots a_n)v_1v_1^* + (a_1 \cdots a_{n-1}a_n^{-1})v_2v_2^*. \end{aligned}$$

Hence, for any diagonal $x \in X^{op}$, we have

$$\begin{aligned} D(x) &= \begin{pmatrix} d_n(x) & 0 \\ 0 & d_n(x)^{-1}d_{n-1}(x)^2 \end{pmatrix} \\ &= d_{n-1}(x) \begin{pmatrix} d_n(x)d_{n-1}(x)^{-1} & 0 \\ 0 & d_n(x)^{-1}d_{n-1}(x) \end{pmatrix}. \end{aligned} \quad (2.16)$$

For any $x \in X^{op}$, we have $x = b \cdot y$ for some diagonal $y \in X^{op}$ and $b \in B$. Then, by (2.15),

$$D(x)[\rho(p)v] = D(b \cdot y)[\rho(p)v] = \psi_{n-1}(y)D(y)[v],$$

hence, by (2.16),

$$\begin{aligned} D(x) &= \psi_{n-1}(y) (\rho(b)^{*-1} \cdot D(y)) = \psi_{n-1}(y)d_{n-1}(y) (\rho(b)^{*-1} \cdot D_1(y)) \\ &= d_{n-1}(x) (\rho(b)^{*-1} \cdot D_1(y)), \end{aligned}$$

and $D_1(x) = \rho(b)^{*-1} \cdot D_1(y) \in X_1$, since $\rho(b)^{*-1} \in G_1 = U(j_2)$. \square

Proof of Theorem 2.3. By the embedding

$$K_1 = U(j_2) \hookrightarrow K = K_n, \quad h \mapsto \tilde{h} = \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix}, \quad (2.17)$$

we see

$$\begin{aligned} \omega(x; s) &= \int_{K_1} dh \int_K |\mathbf{d}(k \cdot x)|^{s+\varepsilon} dk \\ &= \int_{K_1} dh \int_K |\mathbf{d}(\tilde{h}k \cdot x)|^{s+\varepsilon} dk \\ &= \int_K \prod_{i < n} |d_i(k \cdot x)|^{s_i+\varepsilon_i} \int_{K_1} |d_n(\tilde{h}k \cdot x)|^{s_n+\varepsilon_n} dh dk. \end{aligned} \quad (2.18)$$

For $y \in X^{op}$, we have

$$\begin{aligned} d_n(\tilde{h} \cdot y) &= g(\tilde{h} \cdot y, v_0) = g((\tilde{h}, 1) \star (y, h^{-1}v_0)) \\ &= g(y, h^{-1}v_0) \quad (\text{since } \psi_{n-1}(\tilde{h}) = 1) \\ &= D(y)[h^{-1}v_0] = \hat{d}_1(h^{*-1} \cdot D(y)) = d_{n-1}(y)\hat{d}_1(h^{*-1} \cdot D_1(y)), \end{aligned}$$

where $\hat{d}_1(\cdot)$ is the $(1, 1)$ -component of \cdot . Since $\hat{d}_1(x_1) = d_1(x_1^{-1})$ for $x_1 \in X_1$, we have

$$d_n(\tilde{h} \cdot y) = d_{n-1}(y)d_1((h^{*-1} \cdot D_1(y))^{-1}) = d_{n-1}(y)d_1(h \cdot D_1(y)^{-1}).$$

Returning to (2.18), we have

$$\begin{aligned} \omega(x; s) &= \int_K \prod_{i < n} |d_i(k \cdot x)|^{s_i+\varepsilon_i} \int_{K_1} |d_{n-1}(k \cdot x)d_1(h \cdot D_1(k \cdot x)^{-1})|^{s_n+\varepsilon_n} dh dk \\ &= \int_K \prod_{i < n} |d_i(k \cdot x)|^{s_i+\varepsilon_i} |d_{n-1}(k \cdot x)|^{s_n+\varepsilon_n} \omega^{(1)}(D_1(k \cdot x)^{-1}; s_n) dk, \end{aligned}$$

where $\omega^{(1)}(y; s)$ is the spherical function of size $n = 1$. Then, by Proposition 2.4, we obtain

$$\omega(x; s) = \omega(x; s_1, \dots, s_{n-2}, s_{n-1} + 2s_n, -s_n),$$

which shows in z -variable

$$\omega(x; z) = \omega(x; \tau(z)), \quad \tau(z) = (z_1, \dots, z_{n-1}, -z_n),$$

and we conclude the proof. \square

2.3. Since our spherical function $\omega(x; s)$ satisfies the same functional equations with respect to S_n and τ (Theorem 2.1 and Theorem 2.3) as $\omega_T(x; s)$ in [Hir11] does, we have the same functional equations with respect to W also. For the proof, we may follow a similar line as in [Hir11, §2.3], so we omit the details.

We denote by Σ the set of roots of G with respect to the maximal k -split torus of G contained in B and by Σ^+ the set of positive roots with respect to B . We may understand Σ as a subset in \mathbb{Z}^n , and set

$$\begin{aligned} \Sigma^+ &= \Sigma_s^+ \cup \Sigma_\ell^+, \\ \Sigma_s^+ &= \{e_i - e_j, e_i + e_j \mid 1 \leq i < j \leq n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\}, \end{aligned} \tag{2.19}$$

where e_i is the i -th unit vector in \mathbb{Z}^n , $1 \leq i \leq n$. For each $\sigma \in W$, we set

$$\Sigma_s^+(\sigma) = \{\alpha \in \Sigma_s^+ \mid -\sigma(\alpha) \in \Sigma^+\}. \tag{2.20}$$

We define a pairing on $\mathbb{Z}^n \times \mathbb{C}^n$ by

$$\langle t, z \rangle = \sum_{i=1}^n t_i z_i, \quad (t \in \mathbb{Z}^n, z \in \mathbb{C}^n),$$

which satisfies

$$\langle \alpha, z \rangle = \langle \sigma(\alpha), \sigma(z) \rangle, \quad (\alpha \in \Sigma, z \in \mathbb{C}^n, \sigma \in W).$$

THEOREM 2.6. *The spherical function $\omega(x; z)$ satisfies the following functional equation*

$$\omega(x; z) = \Gamma_\sigma(z) \cdot \omega(x; \sigma(z)), \tag{2.21}$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma_s^+(\sigma)} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}},$$

and we understand $\Gamma_\sigma(z) = 1$ if $\Sigma_s^+(\sigma) = \emptyset$.

The next theorem can be proved in a similar line to the proof of [Hir11, Theorem 2.9].

THEOREM 2.7. *The function $G(z) \cdot \omega(x; z)$ is holomorphic on \mathbb{C}^n and W -invariant, in particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$, where*

$$G(z) = \prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle - 1}}.$$

An outline of a proof. It is clear that $G(z) \cdot \omega(x; z)$ is invariant under the action of $\sigma \in \{(i, i+1) \in S_n \mid 1 \leq i \leq n-1\} \cup \{\tau\}$ by Theorem 2.6, hence it is invariant for every $\sigma \in W$ by cocycle relations of Gamma factors.

In order to prove the holomorphy, we consider the following integral for any compactly supported function φ in $\mathcal{C}^\infty(K \backslash X)$:

$$\Phi(z; \varphi) = \int_{X^{op}} \varphi(x) |\mathbf{d}(x)|^{s+\varepsilon} dx,$$

where dx is a G -invariant measure on X and the relation of z and s and ε are the same as before. Then, the right hand side is absolutely convergent if $\operatorname{Re}(s_i) \geq 1$, $(1 \leq i \leq n-1)$ and $\operatorname{Re}(s_n) \geq \frac{1}{2}$, and continued to a rational function on q^{s_1}, \dots, q^{s_n} . Taking φ to be the characteristic function of $K \cdot x$, we see $\Phi(z; \varphi) = \operatorname{vol}(K \cdot x) \cdot \omega(x; z)$, hence it is enough to show the holomorphy of $G(z) \cdot \Phi(z; \varphi)$. To begin with, $\Phi(z; \varphi)$ is holomorphic on

$$\mathcal{D}_0 = \left\{ z \in \mathbb{C}^n \mid -\frac{1}{2} \geq \operatorname{Re}(z_n), \operatorname{Re}(z_{i+1}) \geq \operatorname{Re}(z_i) + 1, (1 \leq i \leq n-1) \right\},$$

and satisfies the same functional equations

$$\Phi(z; \varphi) = \Gamma_\sigma(z) \Phi(\sigma(z); \varphi), \quad \sigma \in W.$$

Since $G(z)$ is holomorphic on \mathcal{D}_0 , $G(z) \cdot \Phi(z; \varphi)$ is holomorphic on

$$\bigcup_{\sigma \in W} \sigma(\mathcal{D}_0).$$

We recall $G_1(z)$ in Theorem 2.1 and write

$$G(z) = G_1(z) \times G_2(z), \quad G_2(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i + z_j}}{1 - q^{z_i + z_j - 1}}.$$

We obtain, in a similar way to the proof of Theorem 2.1,

$$\Phi(z; \varphi) = \int_{X^{op}} \varphi(x) \zeta_*^{(h)}(D(x); z) dx,$$

then we see $G(z) \cdot \Phi(z; \varphi)$ is holomorphic on

$$\mathcal{D}_1 = \{z \in \mathbb{C}^n \mid \operatorname{Re}(z_i + z_j) \neq 1, (1 \leq i < j \leq n)\},$$

since $G_2(z)$ is holomorphic on \mathcal{D}_1 and φ is compactly supported. We see $G(z) \cdot \Phi(z; \varphi)$ is holomorphic on \mathbb{C}^n , since it is holomorphic on

$$\tilde{\mathcal{D}} = \bigcup_{\sigma \in W} \sigma(\mathcal{D}_0 \cup \mathcal{D}_1),$$

and the convex hull of the connected set $\tilde{\mathcal{D}}$ is \mathbb{C}^n . □

3. The explicit formula for $\omega(x; z)$

3.1. We give the explicit formula of $\omega(x; z)$. Since $\omega(x; z)$ is stable on each K -orbit, it is enough

to show the explicit formula for each x_λ , $\lambda \in \Lambda_n^+$ by Theorem 1.1.

THEOREM 3.1. *For $\lambda \in \Lambda_n^+$, one has the explicit formula:*

$$\omega(x_\lambda; z) = \frac{(1 - q^{-2})^n}{w_{2n}(-q^{-1})} \cdot \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}} \cdot c_\lambda \cdot Q_\lambda(z), \quad (3.1)$$

where

$$\begin{aligned} w_m(t) &= \prod_{i=1}^m (1 - t^i), \quad c_\lambda = (-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})}, \\ Q_\lambda(z) &= \sum_{\sigma \in W} \sigma \left(q^{-\langle \lambda, z \rangle} c(z) \right), \\ c(z) &= \prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}} \prod_{\alpha \in \Sigma_\ell^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}. \end{aligned} \quad (3.2)$$

Remark 3.2. The above formula is the same as the explicit formula of $\omega_T(y_\lambda; z)$ on X_T at $y_\lambda \in X_T$ parametrized by $\lambda \in \Lambda_n^+$ in [Hir11, Theorem 3.3]. We explain the relation between the spaces X and X_T 's in Appendix C.

We see that the main part $Q_\lambda(z)$ of $\omega(x_\lambda; z)$ belongs to $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$ by Theorem 2.7. On the other hand $Q_\lambda(z)$ is a Hall-Littlewood polynomial P_λ of type C_n up to constant multiple, which is introduced in a general context of orthogonal polynomials associated with root systems ([Mac00, §10]), and $Q_0(z)$ is a specialization of Poincaré polynomial ([Mac72, Th.2.8]). More precisely,

$$\begin{aligned} Q_\lambda(z) &= \frac{\widetilde{w}_\lambda(-q^{-1})}{(1 + q^{-1})^n} \cdot P_\lambda(z), \\ \widetilde{w}_\lambda(t) &= w_{m_0(\lambda)}(t)^2 \cdot \prod_{\ell \geq 1} w_{m_\ell(\lambda)}(t), \quad m_\ell(\lambda) = \#\{i \mid \lambda_i = \ell\}, \end{aligned} \quad (3.3)$$

and it is known that the set $\{Q_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms a \mathbb{C} -basis for \mathcal{R} , and in particular, $Q_0(z)$ is a constant independent of z . For details see Appendix B.

By Theorem 3.1 and Remark 3.2, we have the following corollary.

COROLLARY 3.3. *For $x_0 = 1_{2n}$, one has*

$$\omega(1_{2n}; z) = \frac{(1 - q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \times \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}}.$$

Proof. By definition, we see that $\omega(x; -\varepsilon) = 1$ with s -variable $-\varepsilon$. We denote by z^* the value in z -variable corresponding to $-\varepsilon$. Since $Q_0(z)$ is independent of z ,

$$Q_0(z) = Q_0(z^*) = \left\{ \frac{(1 - q^{-2})^n}{w_{2n}(-q^{-1})} \cdot \frac{1}{G(z^*)} \right\}^{-1} = \frac{w_n(-q^{-1})^2}{(1 + q^{-1})^n},$$

and

$$\begin{aligned}\omega(1_{2n}; z) &= \omega(x_0; z) = \frac{(1 - q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \cdot \frac{1}{G(z)} \\ &= \frac{(1 - q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}}.\end{aligned}$$

□

We will prove Theorem 3.1 by using a general expression formula given in [Hir10] (or in [Hir99]) of spherical functions on homogeneous spaces, which is based on functional equations of finer spherical functions and some data depending only on the group G . We need to check the assumptions there. Let \mathbb{G} be a connected reductive linear algebraic group and \mathbb{X} be a \mathbb{G} -homogeneous affine algebraic variety, where everything is assumed to be defined over a p -adic field k . For an algebraic set, we use the same ordinary letter to indicate the set of k -rational points. Let K be a special good maximal compact open subgroup of G , and \mathbb{B} a minimal parabolic subgroup of \mathbb{G} defined over k satisfying $G = KB = BK$. We denote by $\mathfrak{X}(\mathbb{B})$ the group of rational characters of \mathbb{B} defined over k and by $\mathfrak{X}_0(\mathbb{B})$ the subgroup consisting of those characters associated with some relative \mathbb{B} -invariant on \mathbb{X} defined over k . In these situation, the assumptions are the following:

- (A1) \mathbb{X} has only a finite number of \mathbb{B} -orbits (, hence there is only one open orbit \mathbb{X}^{op}).
- (A2) A basic set of relative \mathbb{B} -invariants on \mathbb{X} defined over k can be taken by regular functions on \mathbb{X} .
- (A3) For $y \in \mathbb{X} \setminus \mathbb{X}^{op}$, there exists some ψ in $\mathfrak{X}_0(\mathbb{B})$ whose restriction to the identity component of the stabilizer \mathbb{B}_y of \mathbb{B} at y is not trivial.
- (A4) The rank of $\mathfrak{X}_0(\mathbb{B})$ coincides with that of $\mathfrak{X}(\mathbb{B})$.

In the present situation, our space X is isomorphic to $U(j_{2n})/U(1_{2n})$ over \bar{k} (cf. Appendix A and (1.2)), which is a symmetric space and (A1) is satisfied. (A2) and (A4) are satisfied by our relative B -invariants $\{d_i(x) \mid 1 \leq i \leq n\}$, where n is the rank of $\mathfrak{X}_0(\mathbb{B}) = \mathfrak{X}_0(\mathbb{B})$ and $\mathbb{X}^{op} = \{x \in \mathbb{X} \mid d_i(x) \neq 0, 1 \leq i \leq n\}$. To check (A3) is crucial and rather complicated.

We admit the condition (A3) for a while, which is proved in §3.2, and prove Theorem 3.1. The set $X^{op} = \{x \in X \mid d_i(x) \neq 0, 1 \leq i \leq n\}$ is decomposed into the disjoint union of B -orbits as follows:

$$\begin{aligned}X^{op} &= \bigsqcup_{u \in \mathcal{U}} X_u, \quad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^n, \\ X_u &= \{x \in X^{op} \mid v_\pi(d_i(x)) \equiv u_1 + \cdots + u_i \pmod{2}, 1 \leq i \leq n\}.\end{aligned}$$

According to the decomposition of X^{op} , we consider finer spherical functions

$$\omega_u(x; s) = \int_K |\mathbf{d}(k \cdot x)|_u^{s+\varepsilon} dk, \quad |\mathbf{d}(y)|_u^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X_u, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for “generic z ”, the set $\{\omega_u(x, z) \mid u \in \mathcal{U}\}$ becomes a basis for the space of spherical functions on X associated with the same λ_z , where we keep the relation (2.8) between s and z . Here “generic z ” means that $f(q^{z_1}, \dots, q^{z_n}) \neq 0$ for a polynomial $f(x_1, \dots, x_n)$, which comes from the bijectivity of Poisson integral (cf. [Kat81, Theorem 3.2]) and the condition (A3) (cf. [Hir10, Lemma 2.4]). For each character χ of \mathcal{U} , we may represent as follows

$$\sum_{u \in \mathcal{U}} \chi(u) \omega_u(x; z) = \omega(x; z_\chi), \quad (3.4)$$

where z_χ is obtained by adding $\frac{\pi\sqrt{-1}}{\log q}$ to z_i for suitable i according to χ , and they are linearly independent (for generic z) as varying characters χ . By the functional equation of $\omega(x; z)$ (Theorem 2.6), we have for each $\sigma \in W$

$$\begin{aligned} \omega(x; z_\chi) &= \Gamma_\sigma(z_\chi) \omega(x; \sigma(z_\chi)) \\ &= \Gamma_\sigma(z_\chi) \omega(x; \sigma(z)_{\sigma(\chi)}), \end{aligned} \quad (3.5)$$

by taking a suitable character $\sigma(\chi)$ of \mathcal{U} . When χ is the trivial character $\mathbf{1}$, the equation (3.5) coincides with the original functional equation of $\omega(x; z)$ and $\Gamma_\sigma(z_{\mathbf{1}}) = \Gamma_\sigma(z)$. By (3.4) and (3.5), we obtain vector-wise functional equations for finer spherical functions $\omega_u(x; z)$

$$(\omega_u(x; z))_{u \in \mathcal{U}} = A^{-1} \cdot G(\sigma, z) \cdot \sigma A \cdot (\omega_u(x; \sigma(z)))_{u \in \mathcal{U}}, \quad \sigma \in W, \quad (3.6)$$

where

$$A = (\chi(u))_{\chi, u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi, u} \in GL_{2^n}(\mathbb{Z}),$$

χ runs over characters of \mathcal{U} , $u \in \mathcal{U}$, and $G(\sigma, z)$ is the diagonal matrix of size 2^n whose (χ, χ) -component is $\Gamma_\sigma(z_\chi)$. Here we fix the first entry of χ to be the trivial character $\mathbf{1}$. We denote by U the Iwahori subgroup of K compatible with B and take the normalized Haar measure du on U ;

$$U = \left\{ \nu = (u_{ij}) \in K \mid \begin{array}{l} u_{ii} \in \mathcal{O}_{k'}^\times \text{ for } 1 \leq i \leq 2n, \\ u_{ij} \in \pi \mathcal{O}_{k'} \text{ if } i > j \end{array} \right\}.$$

Then it is easy to see, for any $\nu \in U$ and x_λ with $\lambda \in \Lambda_n^+$

$$|d_i(\nu \cdot x_\lambda)| = \left| d_i(\text{Diag}(\pi^{-\lambda_n}, \dots, \pi^{-\lambda_1})) \right| = q^{(\lambda_1 + \dots + \lambda_i)}$$

which means $x_\lambda \in \mathcal{R}^+$ in the sense of [Hir10, (2.8)]. We set

$$\delta_u(x_\lambda, z) = \int_U |\mathbf{d}(\nu \cdot x_\lambda)|_u^{s+\varepsilon} d\nu.$$

Then we have

$$\delta_u(x_\lambda, z) = \begin{cases} |\mathbf{d}(x_\lambda)|^{s+\varepsilon} & \text{if } x_\lambda \in X_u \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} c_\lambda q^{-\langle \lambda, z \rangle} & \text{if } x_\lambda \in X_u \\ 0 & \text{otherwise.} \end{cases} \quad (3.7)$$

Applying [Hir10, Theorem 2.6] to our present case, we obtain for generic z , by virtue of (3.6),

$$(\omega_u(x_\lambda; z))_{u \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\delta_u(x_\lambda, \sigma(z)))_{u \in \mathcal{U}}, \quad (3.8)$$

where

$$\begin{aligned} Q &= \sum_{\sigma \in W} [U\sigma U : U]^{-1} = \frac{w_{2n}(-q^{-1})}{(1-q^{-2})^2}, \\ \gamma(z) &= \prod_{\alpha \in \Sigma_s^+} \frac{1-q^{2\langle \alpha, z \rangle - 2}}{1-q^{2\langle \alpha, z \rangle}} \cdot \prod_{\alpha \in \Sigma_\ell^+} \frac{1-q^{\langle \alpha, z \rangle - 1}}{1-q^{\langle \alpha, z \rangle}}. \end{aligned} \quad (3.9)$$

Then we obtain

$$\begin{aligned} \omega(x_\lambda; z) &= \sum_{u \in \mathcal{U}} \mathbf{1}(u) \omega_u(x_\lambda; z) \\ &= \text{the first entry of } A(\omega_u(x_\lambda; z))_{u \in \mathcal{U}} \\ &= \text{the first entry of } \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) (G(\sigma, z) \cdot \sigma A) (\delta_u(x_\lambda, \sigma(z)))_{u \in \mathcal{U}} \\ &= \frac{(1-q^{-2})^n}{w_{2n}(-q^{-1})} \times \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_\sigma(z) \sum_u \delta_u(x_\lambda, \sigma(z)) \\ &= \frac{c_\lambda(1-q^{-2})^n}{w_{2n}(-q^{-1})} \times \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_\sigma(z) q^{-\langle \lambda, \sigma(z) \rangle}. \end{aligned}$$

By Theorem 2.6, Theorem 2.7, (3.2) and (3.9), we have

$$\Gamma_\sigma(z) = \frac{G(\sigma(z))}{G(z)}, \quad \gamma(z) \cdot G(z) = c(z).$$

Thus we obtain the required explicit formula of $\omega(x_\lambda; z)$ for generic z , and it is also valid for any $z \in \mathbb{C}^n$, since $G(z) \cdot \omega(x_\lambda; z)$ is a polynomial in $q^{\pm z_1}, \dots, q^{\pm z_n}$. \square

3.2. In this subsection, we prove the space X_n satisfies the condition (A3) by induction on n . For $n = 1$, the condition (A3) is obvious, since $X_1 = X_1^{op}$ (cf. Proposition 1.2). Hereafter we assume $n \geq 2$. We set

$$\begin{aligned} t(\mathbf{b}) &= \text{Diag}(b_1, \dots, b_n, b_n^{-1}, \dots, b_1^{-1}) \in B (= B_n), \quad \mathbf{b} = (b_1, \dots, b_n) \in (k^\times)^n, \\ B_0 &= \left\{ \begin{pmatrix} j_n b^{*-1} j_n & 0 \\ 0 & b \end{pmatrix} \in B \mid b \in GL_n(k'), \text{ upper triangular} \right\}, \\ N_B &= \left\{ \begin{pmatrix} 1_n & A \\ 0 & 1_n \end{pmatrix} \in B \mid jA + A^*j = 0 \right\}. \end{aligned}$$

LEMMA 3.4. Assume $x \in X_n$ and $d_1(x) \neq 0$. Then, the orbit $B \cdot x$ contains an element of type

$$\left(\begin{array}{c|c|c} a^{-1} & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & a \end{array} \right), \quad a = 1, \pi, \quad y \in X_{n-1}.$$

Here, if $x \notin X_n^{op}$, then $y \notin X_{n-1}^{op}$.

Proof. By the action of B_0 , we may assume $x = (x_{i,j})$ satisfies

$$x_{2n,2n} = 1, \pi, \quad x_{2n,j} = x_{j,2n} = 0, \quad (n+1 \leq j \leq 2n-1).$$

Then, by the action of N_B , we may change

$$x_{2n,j} = x_{j,2n} = 0, \quad (2 \leq j \leq n), \quad x_{2n,1} = x_{1,2n} \in k.$$

Then, by the property $x^* = x$ and $j_{2n}[x] = j_{2n}$, we see

$$x_{1,j} = x_{j,1} = 0, \quad (j \geq 2), \quad x_{1,1} = x_{2n,2n}^{-1},$$

hence may assume x has the shape as in the statement, and $\Phi_{xj_{2n}}(t) = (t^2 - 1)\Phi_{yj_{2n-2}}(t)$, hence $y \in X_{n-1}$. The second assertion is clear. \square

LEMMA 3.5. Assume $x \in X_n$, $d_1(x) = 0$ and the first non-zero entry in the $2n$ -th column from the bottom stands at $(2n - \ell + 1, 2n)$ with $1 < \ell \leq n$. Then there is some $y = (y_{i,j}) \in B \cdot x$ which satisfies

$$\begin{aligned} y_{1,j} &= y_{j,1} = \delta_{j,\ell}, & y_{2n,j} &= y_{j,2n} = \delta_{2n-\ell+1,j}, \\ y_{\ell,j} &= y_{j,\ell} = \delta_{1,j}, & y_{2n-\ell+1,j} &= y_{j,2n-\ell+1} = \delta_{2n,j}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta.

The stabilizer B_y contains $t(\mathbf{b})$ with $b_1 = b_\ell^{-1} = b \in k^\times$ and the remaining b_i being 1, and the character ψ_1 is not trivial on B_y .

Proof. By the action of B_0 , we may assume $x = (x_{i,j})$ satisfies

$$\begin{aligned} x_{2n-\ell+1,j} &= x_{j,2n-\ell+1} = \begin{cases} 1 & \text{if } j = 2n \\ 0 & \text{if } n+1 \leq j \leq 2n-1, \end{cases} \\ x_{2n,j} &= x_{j,2n} = \begin{cases} 1 & \text{if } j = 2n - \ell + 1 \\ 0 & \text{if } n+1 \leq j \leq 2n, j \neq 2n - \ell + 1. \end{cases} \end{aligned}$$

Then by the action of N_B , we may take $y \in B \cdot x$ such that

$$\begin{aligned} y_{2n,j} &= y_{j,2n} = \begin{cases} 1 & \text{if } j = 2n - \ell + 1 \\ a & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell, 2n - \ell + 1, \end{cases} \\ y_{2n-\ell+1,j} &= \overline{y_{j,2n-\ell+1}} = \begin{cases} 1 & \text{if } j = 2n \\ b & \text{if } j = 1, \\ c & \text{if } j = \ell, \\ 0 & \text{if } j \neq 1, \ell, 2n, \end{cases} \end{aligned}$$

where $a \in k$ and $b, c \in k'$. Then, by the property $y = y^*$ and $j_{2n}[y] = j_{2n}$, we see y has the required shape, and it is clear the stabilizer B_y contains the elements in the statement. \square

LEMMA 3.6. Assume $x \in X_n$, $d_1(x) = 0$ and the first non-zero entry in the $2n$ -th column from the bottom stands at $(\ell, 2n)$ with $1 < \ell \leq n$. Then there is some $y = (y_{i,j}) \in B \cdot x$ which satisfies

$$\begin{aligned} y_{1,j} &= y_{j,1} = \delta_{j,2n-\ell+1}, & y_{2n,j} &= y_{j,2n} = \delta_{\ell,j}, \\ y_{\ell,j} &= y_{j,\ell} = \delta_{2n,j}, & y_{2n-\ell+1,j} &= y_{j,2n-\ell+1} = \delta_{1,j}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta.

The stabilizer B_y contains $t(\mathbf{b})$ with $b_1 = b_\ell = b \in k^\times$ and the remaining b_i being 1, and the character ψ_1 is not trivial on B_y .

Proof. By the action of B , we may assume $x = (x_{i,j})$ satisfies $x_{2n,j} = x_{j,2n} = \delta_{j,\ell}$. Then, by the action of B , we may take $y \in B \cdot x$ such that

$$y_{2n,j} = \overline{y_{j,2n}} = \begin{cases} a & \text{if } j = 1 \\ 1 & \text{if } j = \ell \\ 0 & \text{if } j \neq 1, \ell, \end{cases}$$

$$y_{\ell,j} = \overline{y_{j,\ell}} = \begin{cases} b & \text{if } j = 1 \\ 1 & \text{if } j = n \\ 0 & \text{if } j \neq 1, n, \end{cases}$$

where $a, b \in k'$. Then, by the property $y^* = y$ and $j_{2n}[y] = j_{2n}$, we see y has the required shape, and it is clear that the stabilizer B_y contained the elements in the statement. \square

LEMMA 3.7. Assume $x \in X_n$, $d_1(x) = 0$ and $x_{2n,i} = 0$ for $2 \leq i \leq 2n$. Then x has the following shape:

$$x = \left(\begin{array}{c|c|c} * & * & \xi \\ \hline & & 0 \\ * & y & \vdots \\ \hline & & 0 \\ \xi & 0 \cdots 0 & 0 \end{array} \right), \quad \xi = \pm 1, y \in \tilde{X}_{n-1}, \prod_{i=1}^{n-1} d_i(y) = 0.$$

Proof. Since $j_{2n}[x] = j_{2n}$, we see $x_{2n,1}^2 = 1$ and x has the shape written as above and $y = y^* \in \tilde{X}_{n-1}$. If y was diagonalizable by the action of B_{n-1} , then $\Phi_{xj_{2n}}(t) = (t^2 - 1)^{n-1}(t - \xi)^2$, which contradicts to the fact $x \in X_n$; hence y cannot be diagonalizable and $\prod_{i=1}^{n-1} d_i(y) = 0$. \square

LEMMA 3.8. Assume that $x \in X_n$ has the following shape:

$$x = \left(\begin{array}{c|c|c} a & * & x_r \\ \hline * & y & 0 \\ x_r^* & 0 & 0 \end{array} \right), \quad x_r = \begin{pmatrix} * & & \xi_1 \\ & \ddots & \\ \xi_r & & 0 \end{pmatrix}, \quad y \in \tilde{X}_{n-r}.$$

(i) Any anti-diagonal entry of x_r equals to ± 1 , and under the B_n -action, we may change x_r into dj_r with $d = \text{Diag}(\xi_1, \dots, \xi_r)$.

(ii) Assume further $r = n$. Then r is even, the numbers of $+1$ and -1 within $\{\xi_i \mid 1 \leq i \leq n\}$ are the same, and $a_{ij} = 0$ if $\xi_i = \xi_j$. The stabilizer B_x contains $t(\mathbf{b})$ such that

$$b_i = \begin{cases} b & \text{if } \xi_i = 1 \\ b^{-1} & \text{if } \xi_i = -1 \end{cases} \quad (b \in k^\times),$$

and the character ψ_1 is not trivial on B_x .

Proof. (i) We prove by induction r . It is clear for $r = 1$. We assume the assertion holds for r and consider the case $r + 1$. Then we may assume that the upper right block x_{r+1} can be written as

$$x' = \left(\begin{array}{c|c|c} a_1 & 0 & \xi_1 \\ \vdots & & \ddots \\ a_r & \xi_r & 0 \\ \hline \xi & 0 & \cdots & 0 \end{array} \right).$$

Since $j_{2n}[x] = j_{2n}$, we have $x'j_{r+1}x' = j_{r+1}$ and $a_i = 0$ if $\xi = \xi_i$. Then setting $b \in B$ as

$$b = \left(\begin{array}{c|c|c} c & 0 & 0 \\ \hline 0 & 1_{2(n-r-1)} & 0 \\ \hline 0 & 0 & j_{r+1}c^{*-1}j_{r+1} \end{array} \right), \quad c = \left(\begin{array}{c|c} & -a_1\xi/2 \\ \hline 1_r & \vdots \\ \hline & -a_r\xi/2 \\ \hline 0 & 1 \end{array} \right),$$

the upper right $(r+1)$ by $(r+1)$ block of $b \cdot x$ becomes $\text{Diag}(\xi_1, \dots, \xi_r, \xi)j_{r+1}$ as required.

(ii) Assume $r = n$. Since $\Phi_{xj_{2n}}(t) = (t^2 - 1)^n$, the numbers of 1 and -1 within $\{\xi_i \mid 1 \leq i \leq n\}$ are the same. Since $ad + da = 0_r$, we see $a_{ij} = 0$ if $\xi_i = \xi_j$, and $t(\mathbf{b})$ the above type is contained in B_x . \square

Now, in order to establish the condition (A3), it suffices to consider x of the following type:

$$x = \left(\begin{array}{c|c|c} * & * & dj_r \\ \hline * & y & 0 \\ \hline j_r d & 0 & 0 \end{array} \right) \in X_{n+r} \setminus X_{n+r}^{op}, \quad y \in \tilde{X}_n, \quad d = \text{Diag}(\xi_1, \dots, \xi_r), \quad \xi_i = \pm 1.$$

Here we may consider the action of B_n on y which keeps the shape of x as above. If $d_1(y) \neq 0$, we may assume y has the shape as in Lemma 3.4. By the action of B_{n+r} we may make every entry of x in the $(r+1)$ -th row and column and $(r+2n)$ -th row and column into 0 except $(r+1, r+1)$ -entry and $(r+2n, r+2n)$ -entry which are nonzero. Removing these rows and columns we have $x' \in X_{n+r-1} \setminus X_{n+r-1}^{op}$ of the same type as above.

Thus we may assume that $d_1(y) = 0$ and y has the shape indicated in Lemma 3.5 or Lemma 3.6. By the action of B_{n+r} , we can make every entry of x , outside of y , in the rows and columns of $(r+1)$ -th, $(r+\ell)$ -th, $(r+2n-\ell+1)$ -th, or $(r+2n)$ -th into 0, while keeping the shape of y . Then, by Lemma 3.5 and Lemma 3.6, we see ψ_{r+1} is not trivial on the stabilizer $(B_{n+r})_x$. \square

4. Spherical Fourier transform and Plancherel formula on $\mathcal{S}(K \backslash X)$

We consider the Schwartz space

$$\mathcal{S}(K \backslash X) = \{ \varphi : X \longrightarrow \mathbb{C} \mid \text{left } K\text{-invariant, compactly supported} \},$$

which is spanned by the characteristic function of $K \cdot x$, $x \in X$, and an $\mathcal{H}(G, K)$ -submodule of $\mathcal{C}^\infty(K \backslash X)$ by the convolution product. We define the modified spherical function

$$\Psi(x; z) = \omega(x; z) / \omega(1_{2n}; z) \in \mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W, \quad (4.1)$$

then by Theorem 3.1, Corollary 3.3, (3.3), we have

$$\Psi(x_\lambda; z) = c_\lambda w_\lambda P_\lambda(z), \quad w_\lambda = \frac{\widetilde{w}_\lambda(-q^{-1})}{w_n(-q^{-1})^2}. \quad (4.2)$$

We define the spherical Fourier transform

$$\begin{aligned} F : \mathcal{S}(K \backslash X) &\longrightarrow \mathcal{R} \\ \varphi &\longmapsto F(\varphi)(z) = \int_X \varphi(x) \Psi(x; z) dx, \end{aligned} \quad (4.3)$$

where dx is a G -invariant measure on X . There is a G -invariant measure on X , since X is a disjoint union of two G -orbits, and G is reductive. We don't need to fix the normalization of dx at this moment, we will determine suitably afterward (cf. Theorem 4.5). We denote by $v(K \cdot x)$ for the volume of $K \cdot x$ by dx . For $\lambda \in \Lambda_n^+$, we denote by ch_λ the characteristic function of $K \cdot x_\lambda$ and by ch_λ the characteristic function of $K \cdot x_\lambda$. Then, for $\lambda \in \Lambda_n^+$

$$F(\text{ch}_\lambda)(z) = v(K \cdot x) \Psi(x_\lambda; z) = c_\lambda w_\lambda v(K \cdot x_\lambda) P_\lambda(z). \quad (4.4)$$

We regard \mathcal{R} as an $\mathcal{H}(G, K)$ -module through the Satake isomorphism

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W = \mathcal{R}_0.$$

THEOREM 4.1. *The spherical Fourier transform F gives an $\mathcal{H}(G, K)$ -module isomorphism*

$$\mathcal{S}(K \backslash X) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W (= \mathcal{R}),$$

where \mathcal{R} is regarded as $\mathcal{H}(G, K)$ -module via λ_z . Especially, $\mathcal{S}(K \backslash X)$ is a free $\mathcal{H}(G, K)$ -module of rank 2^n .

Proof. Since the set $\{\text{ch}_\lambda \mid \lambda \in \Lambda_n^+\}$ forms a \mathbb{C} -basis for $\mathcal{S}(K \backslash X)$ and $\{P_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms a \mathbb{C} -basis for \mathcal{R} (cf. Proposition B.3), F is bijective by (4.4). Hence F is an $\mathcal{H}(G, K)$ -module isomorphism, since we have for $f \in \mathcal{H}(G, K)$ and $\varphi \in \mathcal{S}(K \backslash X)$,

$$\begin{aligned} F(f * \varphi) &= \int_X \int_G f(g) \varphi(g^{-1} \cdot x) \Psi(x; z) dg dx \\ &= \int_G \int_X f(g^{-1}) \varphi(y) \Psi(g \cdot y; z) dy dg = \int_X \varphi(y) \int_G f(g^{-1}) \Psi(g \cdot y; z) dg dy \\ &= \lambda_z(f) \int_X \varphi(y) \Psi(y; z) dy = \lambda_z(f) F(\varphi), \end{aligned}$$

where we use the fact $f(g) = f(g^{-1})$ for $g \in G$. Since we see

$$\mathcal{R} = \mathbb{C}[q^{z_1} + q^{-z_1}, \dots, q^{z_n} + q^{-z_n}]^{S_n}, \quad \mathcal{R}_0 = \mathbb{C}[q^{2z_1} + q^{-2z_1}, \dots, q^{2z_n} + q^{-2z_n}]^{S_n},$$

\mathcal{R} is a free \mathcal{R}_0 -module of rank 2^n , and $\mathcal{S}(K \backslash X)$ is a free $\mathcal{H}(G, K)$ -module of rank 2^n . \square

COROLLARY 4.2. *All the spherical functions on X are parametrized by eigenvalues*

$z \in \left(\mathbb{C} / \frac{2\pi\sqrt{-1}}{\log q} \mathbb{Z} \right)^n / W$ through $\mathcal{H}(G, K) \longrightarrow \mathbb{C}$, $f \longmapsto \lambda_z(f)$. The set $\{\Psi(x; z + u) \mid u \in \{0, \pi\sqrt{-1}/\log q\}^n\}$ forms a basis of the space of spherical functions on X corresponding to z .

Proof. The former assertion is clear, since a spherical function $\Psi \in \mathcal{C}^\infty(K \backslash X)$ satisfies, by definition

$$f * \Psi = \lambda_z(f) \Psi, \quad f \in \mathcal{H}(G, K) \quad (4.5)$$

for some $z \in \mathbb{C}^n$, and λ_z is determined by the class of z in $\left(\mathbb{C} / \frac{2\pi\sqrt{-1}}{\log q} \mathbb{Z} \right)^n / W$.

The above $\Psi(x; z + u)$ are linearly independent spherical function corresponding to the same λ_z (cf. (3.4)). We define a pairing on $\mathcal{S}(K \backslash X) \times \mathcal{C}^\infty(K \backslash X)$ by

$$(\varphi, \Psi) = \int_X \varphi(x) \Psi(x) dx, \quad \varphi \in \mathcal{S}(K \backslash X), \quad \Psi \in \mathcal{C}^\infty(K \backslash X),$$

which satisfies

$$(f * \varphi, \Psi) = (\varphi, f * \Psi), \quad (f \in \mathcal{H}(G, K), \varphi \in \mathcal{S}(K \backslash X), \Psi \in \mathcal{C}^\infty(K \backslash X)). \quad (4.6)$$

Let $\{\varphi_i \mid 1 \leq i \leq 2^n\}$ be a free $\mathcal{H}(G, K)$ -basis of $\mathcal{S}(K \backslash X)$. Assume that $\Psi_j \in \mathcal{C}^\infty(K \backslash X)$, $j \in J$ are spherical functions corresponding to the same λ_z , and set

$$\mathbf{a}_j = (a_{ij})_i \in \mathbb{C}^{2^n}, \quad a_{ij} = \int_X \varphi_i(x) \Psi_j(x) dx.$$

Then, by (4.6) and (4.5), it is easy to see that for $c_j \in \mathbb{C}$

$$\begin{aligned} \sum_{j \in J} c_j \Psi_j = 0 \text{ (in } \mathcal{C}^\infty(K \backslash X)) &\iff (\varphi_i, \sum_{j \in J} c_j \Psi_j) = 0, 1 \leq i \leq 2^n \\ &\iff \sum_{j \in J} c_j \mathbf{a}_j = \mathbf{0} \text{ (in } \mathbb{C}^{2^n}). \end{aligned}$$

Hence, there are at most 2^n linearly independent spherical functions, and $\Psi(x; z + u)$'s form a basis. \square

We introduce the following inner product on \mathcal{R} . Set

$$\mathfrak{a}^* = \left\{ \sqrt{-1} \left(\mathbb{R} / \frac{2\pi}{\log q} \mathbb{Z} \right) \right\}^n,$$

and define a measure $d\mu = d\mu(z)$ on \mathfrak{a}^* by

$$d\mu = \frac{1}{n!2^n} \cdot \frac{w_n(-q^{-1})^2}{(1+q^{-1})^n} \cdot \frac{1}{|c(z)|^2} dz,$$

where $c(z)$ is defined in (3.2) and dz is the Haar measure on \mathfrak{a}^* with $\int_{\mathfrak{a}^*} = 1$. For $P, Q \in \mathcal{R}$, we define

$$\langle P, Q \rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z) \overline{Q(z)} d\mu(z).$$

LEMMA 4.3. For $\lambda, \mu \in \Lambda_n^+$, one has

$$\langle P_\lambda, P_\mu \rangle_{\mathcal{R}} = \langle P_\mu, P_\lambda \rangle_{\mathcal{R}} = \delta_{\lambda, \mu} w_\lambda^{-1}.$$

Proof. Applying Proposition B.3 to our case, Hall-Littlewood polynomials associated with the root system of type C_n with $t_s = -q^{-1}$ and $t_\ell = q^{-1}$, we see

$$\begin{aligned} \langle P_\lambda, P_\mu \rangle_{\mathcal{R}} &= \delta_{\lambda, \mu} \cdot \frac{W_{\mathbf{0}}(-q^{-1}, q^{-1})}{W_\lambda(-q^{-1}, q^{-1})} = \delta_{\lambda, \mu} \cdot \frac{(1+q^{-1})^2}{\widetilde{w}_\lambda(-q^{-1})} \\ &= \delta_{\lambda, \mu} w_\lambda^{-1}, \end{aligned}$$

by (B.9) and (4.2). \square

LEMMA 4.4. For $\lambda, \mu \in \Lambda_n^+$ such that $|\lambda| \equiv |\mu| \pmod{2}$,

$$\frac{v(K \cdot x_\lambda)}{v(K \cdot x_\mu)} = \frac{c_\mu^2 w_\mu}{c_\lambda^2 w_\lambda}. \quad (4.7)$$

Proof. We understand $\Psi(x; z) \in \mathcal{C}^\infty(K \backslash X) = \text{Hom}_{\mathbb{C}}(\mathcal{S}(K \backslash X), \mathbb{C})$ as

$$\Psi(\cdot; z) = \sum_{\xi \in \Lambda_n^+} \Psi(x_\xi; z) \text{ch}_\xi. \quad (4.8)$$

For $f \in \mathcal{H}(G, K)$, we write

$$f * \text{ch}_\xi = \sum_{\nu \in \Lambda_n^+} a_\nu^\xi(f) \text{ch}_\nu, \quad (4.9)$$

where almost all $a_\nu^\xi(f) = 0$. Since $\Psi(x; z)$ is a spherical function associated with $[f \mapsto \tilde{f}(z) = \lambda_z(f)]$ for $f \in \mathcal{H}(G, K)$, we have by (4.8) and (4.9)

$$\begin{aligned} \tilde{f}(z) \Psi(x_\lambda; z) &= (f * \Psi(\cdot; z))(x_\lambda) = \sum_{\xi \in \Lambda_n^+} \Psi(x_\xi; z) (f * \text{ch}_\xi)(x_\lambda) \\ &= \sum_{\xi \in \Lambda_n^+} a_\lambda^\xi(f) \Psi(x_\xi; z), \end{aligned}$$

and by (4.2)

$$c_\lambda w_\lambda \tilde{f}(z) P_\lambda(z) = \sum_{\xi \in \Lambda_n^+} a_\lambda^\xi(f) c_\xi w_\xi P_\xi(z). \quad (4.10)$$

Taking the inner product with $P_\mu(z)$, we have by Lemma 4.3,

$$c_\lambda w_\lambda \left\langle \tilde{f}(z) P_\lambda(z), P_\mu(z) \right\rangle_{\mathcal{R}} = a_\lambda^\mu(f) c_\mu. \quad (4.11)$$

By the compatibility of F with $\mathcal{H}(G, K)$ -action and (4.4), taking the image of F of (4.9) for $\xi = \mu$, we have

$$c_\mu w_\mu v(K \cdot x_\mu) \tilde{f}(z) P_\mu(z) = \sum_{\nu \in \Lambda_n^+} a_\nu^\mu(f) c_\nu w_\nu v(K \cdot x_\nu) P_\nu(z).$$

Taking the inner product with $P_\lambda(z)$, we have by Lemma 4.3,

$$c_\mu w_\mu v(K \cdot x_\mu) \left\langle \tilde{f}(z) P_\mu(z), P_\lambda(z) \right\rangle_{\mathcal{R}} = a_\lambda^\mu(f) c_\lambda v(K \cdot x_\lambda). \quad (4.12)$$

By (4.11) and (4.12), we have for any $f \in \mathcal{H}(G, K)$

$$c_\mu^2 w_\mu v(K \cdot x_\mu) \left\langle \tilde{f}(z) P_\mu(z), P_\lambda(z) \right\rangle_{\mathcal{R}} = c_\lambda^2 w_\lambda v(K \cdot x_\lambda) \left\langle \tilde{f}(z) P_\lambda(z), P_\mu(z) \right\rangle_{\mathcal{R}}. \quad (4.13)$$

Now assume $|\lambda| \equiv |\mu| \pmod{2}$, then $x_\lambda = g \cdot x_\mu$ for some $g \in G$. Taking $f \in \mathcal{H}(G, K)$ as the characteristic function of the double coset KgK , we see $a_\lambda^\mu(f) > 0$ and

$$0 \neq \left\langle \tilde{f}(z) P_\lambda(z), P_\mu(z) \right\rangle_{\mathcal{R}} = \left\langle P_\lambda(z), \tilde{f}(z) P_\mu(z) \right\rangle_{\mathcal{R}} = \left\langle \tilde{f}(z) P_\mu(z), P_\lambda(z) \right\rangle_{\mathcal{R}}.$$

Thus we obtain, by (4.13)

$$c_\mu^2 w_\mu v(K \cdot x_\mu) = c_\lambda^2 w_\lambda v(K \cdot x_\lambda),$$

which completes the proof. \square

We recall that X decomposed into two G -orbits:

$$X = G \cdot x_0 \sqcup G \cdot x_1, \quad x_0 = 1_{2n}, \quad x_1 = \text{Diag}(\pi, 1, \dots, 1, \pi^{-1}).$$

THEOREM 4.5 (Plancherel formula on $\mathcal{S}(K \backslash X)$). *By the normalization of G -invariant measure dx such that*

$$v(K \cdot x_\lambda) = c_\lambda^{-2} w_\lambda^{-1}, \quad \lambda \in \Lambda_n^+, \quad (4.14)$$

one has for any $\varphi, \psi \in \mathcal{S}(K \backslash X)$

$$\int_X \varphi(x) \overline{\psi(x)} dx = \int_{\mathfrak{a}^*} F(\varphi)(z) \overline{F(\psi)(z)} d\mu(z). \quad (4.15)$$

Proof. We may normalize dx on X according to the G -orbits as $v(K \cdot x_0) = 1$ on $G \cdot x_0$ and

$$v(K \cdot x_1) = q^{2n-1} \frac{(1 - (-q^{-1})^n)^2}{1 + q^{-1}} \quad \text{on } G \cdot x_1,$$

then it satisfies (4.14) by Lemma 4.4. It suffices to show the identity (4.15) for ch_λ and ch_μ ($\lambda, \mu \in \Lambda_n^+$). Indeed,

$$\int_X \text{ch}_\lambda(x) \overline{\text{ch}_\mu(x)} dx = \delta_{\lambda, \mu} v(K \cdot x_\lambda) = \delta_{\lambda, \mu} c_\lambda^{-2} w_\lambda^{-1},$$

while

$$\begin{aligned} & \int_{\mathfrak{a}^*} F(\text{ch}_\lambda)(z) \overline{F(\text{ch}_\mu)(z)} d\mu(z) \\ &= c_\lambda w_\lambda v(K \cdot x_\lambda) \cdot c_\mu w_\mu v(K \cdot x_\mu) \langle P_\lambda(z), P_\mu(z) \rangle_{\mathcal{R}} = \delta_{\lambda, \mu} c_\lambda^{-2} w_\lambda^{-1}, \end{aligned}$$

which completes the proof. \square

COROLLARY 4.6 (Inversion formula). *For any $\varphi \in \mathcal{S}(K \backslash X)$,*

$$\varphi(x) = \int_{\mathfrak{a}^*} F(\varphi)(z) \Psi(x; z) d\mu(z), \quad x \in X.$$

Proof. For any $x \in X$, we have

$$\begin{aligned} \varphi(x) &= \frac{1}{v(K \cdot x)} \int_X \varphi(y) \text{ch}_x(y) dy \\ &= \frac{1}{v(K \cdot x)} \int_{\mathfrak{a}^*} F(\varphi)(z) \overline{F(\text{ch}_x)(z)} d\mu(z) \\ &= \int_{\mathfrak{a}^*} F(\varphi)(z) \Psi(x; z) d\mu(z). \end{aligned}$$

\square

Appendix A. The spaces of unitary hermitian matrices

In this subsection, let k' be a quadratic extension of a field k of characteristic 0, and consider hermitian matrices with respect to k'/k . As usual we denote by $A^* \in M_{nm}(k')$ the conjugate transpose of $A \in M_{mn}(A)$.

We fix a natural number m and set $\mathcal{H}_m(k') = \{x \in GL_m(k') \mid x^* = x\}$. We define the unitary group for $x \in \mathcal{H}_m(k')$ by

$$U(x) = \{g \in GL_m(k') \mid x[g] = x\}, \quad \text{where } x[g] = g^* x g = g^* \cdot x, \quad (\text{A.1})$$

and, in particular

$$G = U(j_m) \quad \text{for } j_m = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in M_m.$$

We introduce the space \tilde{X} of unitary hermitian matrices and the G -action as follows:

$$\begin{aligned} \tilde{X} &= \{x \in G \mid x^* = x\} = \{x \in GL_m(k') \mid x = x^*, (xj_m)^2 = 1_m\}, \\ g \cdot x &= gxg^* = gxj_mg^{-1}j_m \quad (g \in G, x \in \tilde{X}). \end{aligned} \quad (\text{A.2})$$

We consider these objects as the sets of k -rational points of algebraic sets defined over k . We denote by \bar{k} the algebraic closure of k and realize $G(\bar{k})$ in $\tilde{\mathbb{G}} = \text{Res}_{k'/k}(GL_m)$ as follows. We understand $\tilde{\mathbb{G}} = GL_m(\bar{k}) \times GL_m(\bar{k})$ with the Galois group $\Gamma = \text{Gal}(\bar{k}/k)$ action defined by

$$\sigma(g_1, g_2) = \begin{cases} (g_1^\sigma, g_2^\sigma) & \text{if } \sigma|_{k'} = id \\ (g_2^\sigma, g_1^\sigma) & \text{if } \sigma|_{k'} = \tau, \end{cases} \quad (\sigma \in \Gamma, (g_1, g_2) \in \tilde{\mathbb{G}}),$$

where $g^\sigma = (g_{ij}^\sigma)$ for $g = (g_{ij}) \in GL_m(\bar{k})$ and $\langle \tau \rangle = \text{Gal}(k'/k)$. Then

$$\begin{aligned} \tilde{\mathbb{G}}(k) &= \{(g, g^\tau) \mid g \in GL_m(k')\}, \\ G(\bar{k}) &= \{(g_1, g_2) \in \tilde{\mathbb{G}} \mid {}^t g_2 j_m g_1 = j_m\} = \{(g, j_m {}^t g^{-1} j_m) \mid g \in GL_m(\bar{k})\}, \end{aligned}$$

and the involution $[g \mapsto g^*]$ on G can be extended as $(g_1, g_2)^* = ({}^t g_2, {}^t g_1)$ on $G(\bar{k})$. Hence we see

$$\begin{aligned} \tilde{X}(\bar{k}) &= \{(x, j_m {}^t x^{-1} j_m) \mid x = j_m x^{-1} j_m \in GL_m(\bar{k})\} \\ &= \{(x, j_m {}^t x^{-1} j_m) \mid x \in GL_m(\bar{k}), (xj_m)^2 = 1_m\}, \end{aligned}$$

and the action of $G(\bar{k})$ on $\tilde{X}(\bar{k})$ can be written as

$$\begin{aligned} (g, j_m {}^t g^{-1} j_m) \star (x, j_m {}^t x^{-1} j_m) &= (g, j_m {}^t g^{-1} j_m)(x, j_m {}^t x^{-1} j_m)(j_m g^{-1} j_m, {}^t g) \\ &= (gxj_m g^{-1} j_m, j_m {}^t g^{-1} j_m {}^t x^{-1} j_m {}^t g) \\ &= (gxj_m g^{-1} j_m, j_m {}^t (gxj_m g^{-1} j_m)^{-1} j_m). \end{aligned}$$

Hence we may identify

$$\begin{aligned} G(\bar{k}) &= GL_m(\bar{k}), \quad \tilde{X}(\bar{k}) = \{x \in G(\bar{k}) \mid (xj_m)^2 = 1_m\}, \\ g \star x &= gxj_m g^{-1} j_m, \quad (g \in G(\bar{k}), x \in \tilde{X}(\bar{k})), \end{aligned} \quad (\text{A.3})$$

where $g \star x = g \cdot x$ if $g \in G$ and $x \in X$ (cf. (A.2)).

PROPOSITION A.1. *The space $\tilde{X}(\bar{k})$ decomposes into $G(\bar{k})$ -orbits as follows:*

$$\tilde{X}(\bar{k}) = \bigsqcup_{i=0}^m \left\{ x \in \tilde{X}(\bar{k}) \mid \Phi_{xj_m}(t) = (t-1)^i (t+1)^{m-i} \right\},$$

where $\Phi_y(t)$ is the characteristic polynomial of the matrix y .

Proof. We consider the following $G(\bar{k})$ -set:

$$\begin{aligned} Y(\bar{k}) &= \{y \in G(\bar{k}) \mid y^2 = 1\}, \\ g \circ y &= gyg^{-1} \quad (g \in G(\bar{k}), y \in Y(\bar{k})). \end{aligned}$$

Then the $G(\bar{k})$ -orbits in $Y(\bar{k})$ are determined by characteristic polynomials as follows

$$Y(\bar{k}) = \bigsqcup_{i=0}^m \{y \in Y(\bar{k}) \mid \Phi_y(t) = (t-1)^i(t+1)^{m-i}\}.$$

Since the map

$$\psi : \tilde{X}(\bar{k}) \longrightarrow Y(\bar{k}), \quad x \longmapsto xj_m$$

is bijective and $G(\bar{k})$ -equivariant, we have the $G(\bar{k})$ -orbit decomposition for $\tilde{X}(\bar{k})$ as required. \square

We take the $G(\bar{k})$ -orbit in $\tilde{X}(\bar{k})$ containing 1_m and set

$$X(\bar{k}) = G(\bar{k}) \star 1_m, \quad X = X(\bar{k}) \cap G = X(\bar{k}) \cap \tilde{X}.$$

By Proposition A.1 and its proof, we see

$$X = \left\{ x \in \tilde{X} \mid \Phi_{xj_m}(t) = \Phi_{j_m}(t) \right\}. \quad (\text{A.4})$$

Appendix B. Root systems and Hall-Littlewood polynomials

In this appendix, we summarize several properties of symmetric polynomials, in particular Hall-Littlewood polynomials, which are understood as special cases of Macdonald polynomials. In Appendix B, we understand $\mathbb{N} = \mathbb{N} \cup \{0\}$.

B.1 Root systems

Let V be an n -dimensional real vector space equipped with inner product $\langle \cdot, \cdot \rangle$. Let Σ be an irreducible reduced root system of rank n in V . We fix a set $\Sigma_0 = \{\alpha_i \mid 1 \leq i \leq n\}$ of simple roots, and denote by Σ^+ the set of all positive roots of Σ with respect to Σ_0 . We denote by $\Lambda_0 = \{\mu_i \mid 1 \leq i \leq n\}$ the set of fundamental weights satisfying

$$\langle \alpha_i^\vee, \mu_j \rangle = \delta_{i,j}, \quad \alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle \in V,$$

and set

$$\text{the weight lattice: } \Lambda = \bigoplus_{i=1}^n \mathbb{Z}\mu_i,$$

$$\text{the set of all dominant weights: } \Lambda^+ = \bigoplus_{i=1}^n \mathbb{N}\mu_i,$$

$$\text{the coroot lattice: } Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee.$$

Let τ_α be the reflection defined by $\tau_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$ for $v \in V$, and $W = W(\Sigma)$ be the Weyl group generated by $\{\tau_\alpha \mid \alpha \in \Sigma\}$.

B.2 Symmetric polynomials

We denote by $\mathbb{C}[\Lambda]$ the group algebra of the lattice Λ , which is spanned by the formal exponentials e^λ with $\lambda \in \Lambda$. The Weyl group acts on $\mathbb{C}[\Lambda]$ via the action on Λ : $\sigma(e^\alpha) = e^{\sigma(\alpha)}$ for $\sigma \in W$ and $\lambda \in \Lambda$. We denote by $\mathbb{C}[\Lambda]^W$ the subalgebra of W -invariant elements in $\mathbb{C}[\Lambda]$. For simplicity of notation we set

$$\mathcal{R} = \mathbb{C}[\Lambda]^W, \quad \mathcal{R}_0 = \mathbb{C}[2\Lambda]^W = \mathcal{R} \cap \mathbb{C}[2\Lambda]. \quad (\text{B.1})$$

For each $\lambda \in \Lambda^+$, we set

$$m_\lambda = \sum_{\nu \in W\lambda} e^\nu, \quad W\lambda = \{\sigma\lambda \in \Lambda \mid \sigma \in W\},$$

then it is easy to see that $\{m_\lambda \mid \lambda \in \Lambda^+\}$ forms a \mathbb{C} -basis for \mathcal{R} . We define a partial order \prec on Λ by

$$\mu \prec \lambda \text{ if and only if } \lambda - \mu \in Q^+, \quad (\text{B.2})$$

where

$$Q^+ = \bigoplus_{i=1}^n \mathbb{N}\alpha_i \subset \Lambda.$$

LEMMA B.1. *For $\lambda, \mu \in \Lambda^+$, there exist some $a_\nu \in \mathbb{N}$ ($\nu \in \Lambda^+$) such that*

$$m_\lambda m_\mu = m_{\lambda+\mu} + \sum_{\substack{\nu \in \Lambda^+ \\ \nu \prec \lambda+\mu}} a_\nu m_\nu.$$

Proof. Let $A = W\lambda + W\mu$ and $\kappa \in A$. The W -invariance of A implies that there exists $\sigma \in W$ such that $\sigma\kappa \in A \cap \Lambda^+$. Write $\nu = \sigma\kappa = \sigma_1\lambda + \sigma_2\mu$ for $\sigma_1, \sigma_2 \in W$. Since $\lambda \succ \sigma_1\lambda$ and $\mu \succ \sigma_2\mu$, we have $\lambda + \mu \succ \sigma_1\lambda + \sigma_2\mu = \nu$. Hence we obtain the result. \square

PROPOSITION B.2. *\mathcal{R} is a free \mathcal{R}_0 -module of rank 2^n . Any family $\{p_\mu \mid \mu \in M\}$ satisfying*

$$p_\mu = m_\mu + \sum_{\substack{\nu \in \Lambda^+ \\ \nu \prec \mu}} b_{\mu\nu} m_\nu, \quad (b_{\mu\nu} \in \mathbb{C}), \quad (\text{B.3})$$

$$M = \left\{ \sum_{i=1}^n c_i \mu_i \mid c_i \in \{0, 1\}, 1 \leq i \leq n \right\}$$

forms a free \mathcal{R}_0 -basis for \mathcal{R} .

Proof. Since Λ^+ is a disjoint union of $\mu + 2\Lambda^+$, ($\mu \in M$), we see $\{p_\mu \mid \mu \in M\}$ forms a \mathcal{R}_0 -basis for \mathcal{R} , by Lemma B.1. \square

B.3 Hall-Littlewood polynomials

We introduce a family of orthogonal polynomials, which is a special case of Macdonald polynomials, or can be regarded as Hall-Littlewood polynomials associated with root systems.

We fix parameters $t_\alpha \in \mathbb{R}$ with $|t_\alpha| < 1$ for each $\alpha \in \Sigma$ such that $t_\alpha = t_\beta$ if $(\alpha, \alpha) = (\beta, \beta)$, hence there are at most two independent parameters among t_α 's. We define Hall-Little polynomial associated with the root system Σ , for $\lambda \in \Lambda^+$,

$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{\sigma \in W} \sigma \left(e^\lambda \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha e^{-\alpha}}{1 - e^{-\alpha}} \right), \quad (\text{B.4})$$

where W_λ is the stabilizer in W at λ , and the Poincaré polynomial of a subgroup W' of W is defined by

$$W'(t) = \sum_{\sigma \in W'} \prod_{\alpha \in \Sigma^+ \cap (-\sigma \Sigma^+)} t_\alpha. \quad (\text{B.5})$$

We define the measure $d\mu = d\mu(z)$ on \mathfrak{a}^* as follows.

$$\begin{aligned} \mathfrak{a}^* &= \sqrt{-1} \left(V / \frac{2\pi}{\log q} Q^\vee \right), \quad \int_{\mathfrak{a}^*} dz = 1, \\ d\mu &= d\mu(z) = \frac{W(t)}{\sharp W} \cdot \frac{dz}{|c(z)|^2}, \\ c(z) &= \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}}, \end{aligned} \quad (\text{B.6})$$

We regard $e^\lambda \in \mathbb{C}[\Lambda]$ as a function on \mathfrak{a}^* via $e^\lambda(z) = q^{-\langle \lambda, z \rangle}$. Then we have

$$P_\lambda(z) = \frac{1}{W_\lambda(t)} \sum_{\sigma \in W} \sigma \left(q^{-\langle \lambda, z \rangle} c(z) \right), \quad (\text{B.7})$$

and an inner product on \mathcal{R} is defined by

$$\langle P, Q \rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z) \overline{Q(z)} d\mu(z) \quad (\text{B.8})$$

for $P, Q \in \mathcal{R}$. Then it is known that the following holds.

PROPOSITION B.3 [Mac00, §3, §10]. $\{P_\lambda\}_{\lambda \in \Lambda^+} \subset \mathcal{R}$ satisfies the triangularity condition like (B.3) and forms an orthogonal basis of \mathcal{R} with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ with

$$\langle P_\lambda, P_\mu \rangle_{\mathcal{R}} = \delta_{\lambda, \mu} \frac{W(t)}{W_\lambda(t)}.$$

By definition, we see $W(t) = W_0(t)$. In the case of the root system of type C_n , the explicit forms of the Poincaré polynomial $W_\lambda(t)$ and the Hall-Littlewood polynomial P_λ with the special parameters used in the paper are given in (B.9) and in (B.10) respectively.

B.4 Poincaré polynomials of stabilizers and Hall-Littlewood polynomials in the root system of type C_n

PROPOSITION B.4. For $\lambda \in \Lambda^+$, let

$$\Sigma_{0, \lambda} = \{\alpha \in \Sigma_0 \mid \langle \alpha, \lambda \rangle = 0\} = \bigsqcup_k \Phi_k,$$

where each Φ_k is a connected component of the Dynkin diagram corresponding to $\Sigma_{0, \lambda}$. Denote by W_k the Weyl group generated by the reflections for $\alpha \in \Phi_k$. Then $W_\lambda(t) = \prod_k W_k(t)$.

Proof. Since the stabilizer W_λ is generated by the reflections that fix λ (cf. [Hum90, 1.12, Theorem]), we see that W_λ is generated by $\{\sigma_\alpha\}_{\alpha \in \Sigma_\lambda}$, where

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \langle \alpha, \lambda \rangle = 0\}.$$

Since Σ_λ is a root system, it is sufficient to show that $\Sigma_{0, \lambda}$ is a fundamental system of Σ_λ . Write $\alpha = \sum_{i=1}^n c_i \alpha_i \in \Sigma_\lambda$. Then we have all $c_i \geq 0$ or all $c_i \leq 0$. Furthermore

$$0 = \langle \alpha, \lambda \rangle = \sum_{i=1}^n c_i \langle \alpha_i, \lambda \rangle.$$

Since $\langle \alpha_i, \lambda \rangle \geq 0$ for $1 \leq i \leq n$, we have $c_i = 0$ for such i that $\langle \alpha_i, \lambda \rangle \neq 0$. Thus

$$\alpha = \sum_{\alpha_i \in \Sigma_{0,\lambda}} c_i \alpha_i.$$

□

We use the realization of the root system of C_n introduced in (2.19). Then we have

$$\begin{aligned} \Sigma_0 &= \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}, \\ \Lambda_0 &= \left\{ \mu_i = \sum_{k=1}^i e_k \mid 1 \leq i \leq n \right\}, \end{aligned}$$

and we can identify the set Λ^+ of dominant weights with $\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}$. We fix any $\lambda \in \Lambda_n^+$ and understand

$$\lambda = \sum_{i=1}^n \lambda_i e_i = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \mu_i + \lambda_n \mu_n \in \Lambda^+.$$

Then we have the decomposition

$$\Sigma_{0,\lambda} = \{\alpha \in \Sigma_0 \mid \langle \alpha, \lambda \rangle = 0\} = \bigsqcup_{k=0}^{\infty} \Phi_k,$$

where

$$\Phi_k = \begin{cases} \{\alpha_i \in \Sigma_0 \mid \lambda_i = \lambda_{i+1} = k\} & (k > 0) \\ \{\alpha_i \in \Sigma_0 \mid \lambda_i = 0\} & (k = 0). \end{cases}$$

Let $n_k = \#\{i \mid \lambda_i = k\}$. Note that $n_k = \#\Phi_k + 1$ for $k > 0$ and $n_0 = \#\Phi_0$. Then we see that W_k is isomorphic to the Weyl group of type A_{n_k-1} if $k > 0$, and that W_0 is of type C_{n_0} . Setting $W_{A_0}(t_s) = W_{C_0}(t_s, t_\ell) = 1$ formally, we obtain by Proposition B.4

$$W_\lambda(t) = \left(\prod_{k=1}^{\infty} W_{A_{n_k-1}}(t_s) \right) \cdot W_{C_{n_0}}(t_s, t_\ell)$$

where t_s and t_ℓ are attached to short roots and long roots respectively.

The Poincaré polynomials of type A_{n-1} and of type C_n are respectively given as

$$\begin{aligned} W_{A_{n-1}}(t) &= \prod_{i=1}^{n-1} \frac{1 - t^{i+1}}{1 - t}, \\ W_{C_n}(t_s, t_\ell) &= \prod_{i=0}^{n-1} (1 + t_s^i t_\ell) \frac{1 - t_s^{i+1}}{1 - t_s}. \end{aligned}$$

In the case $t_s = -q^{-1}$ and $t_\ell = q^{-1}$, we obtain

$$\begin{aligned} W_{A_{n-1}}(-q^{-1}) &= \frac{w_n(-q^{-1})}{(1 + q^{-1})^n}, \\ W_{C_n}(-q^{-1}, q^{-1}) &= \frac{w_n(-q^{-1})^2}{(1 + q^{-1})^n}. \end{aligned}$$

Finally we arrive at the explicit form of the Poincaré polynomials and the Hall-Littlewood poly-

nomials in the root system of type C_n with $t_s = -q^{-1}$ and $t_\ell = q^{-1}$ as follows.

$$W_\lambda(-q^{-1}, q^{-1}) = \frac{\widetilde{w}_\lambda(-q^{-1})}{(1 + q^{-1})^n}, \quad (\text{B.9})$$

$$\begin{aligned} P_\lambda(z) &= \frac{(1 + q^{-1})^n}{\widetilde{w}_\lambda(-q^{-1})} \cdot \sum_{\sigma \in W} \sigma \left(q^{-\langle \lambda, z \rangle} c(z) \right), \\ c(z) &= \prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}} \prod_{\alpha \in \Sigma_\ell^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}. \end{aligned} \quad (\text{B.10})$$

Appendix C. Relation with the space X_T

We assume that k'/k be an unramified quadratic extension of \mathfrak{p} -adic fields. In this subsection we explain the relevance of the present space $X = X_n$ to the spaces introduced in [Hir11], and show the expectation there is correct for odd residual characteristic case.

In [Hir11], we have considered for each $T \in \mathcal{H}_n(k')$

$$X_T = \mathfrak{X}_T / U(T), \quad \mathfrak{X}_T = \{x \in M_{2n,n}(k') \mid H_n[x] = T\}, \quad H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix},$$

where $U(H_n) = \{g \in GL_{2n}(k') \mid H_n[g] = H_n\}$ acts homogeneously on X_T by the left multiplication, and the stabilizer at a point in X_T is isomorphic to $U(T) \times U(T)$ (cf. [Hir11, Lemma 1.1]). We assume T is diagonal and realize X_T as a set of k -rational points in an algebraic set defined over k . We set

$$\mathfrak{X}_T(\bar{k}) = \{(x, y) \in M_{2n,n}(\bar{k}) \oplus M_{2n,n}(\bar{k}) \mid {}^t y H_n x = T\},$$

on which $\Gamma = \text{Gal}(\bar{k}/k)$ acts by

$$\sigma(x, y) = \begin{cases} (x^\sigma, y^\sigma) & \text{if } \sigma|_{k'} = \text{id} \\ (y^\sigma, x^\sigma) & \text{if } \sigma|_{k'} = \tau, \end{cases}$$

where $\langle \tau \rangle = \text{Gal}(k'/k)$. We set

$$\begin{aligned} \mathbb{U}(H_n) &= U(H_n)(\bar{k}) = \{(g_1, g_2) \in GL_{2n}(\bar{k}) \times GL_{2n}(\bar{k}) \mid {}^t g_2 H_n g_1 = H_n\}, \\ \mathbb{U}(T) &= U(T)(\bar{k}) = \{(h_1, h_2) \in GL_n(\bar{k}) \times GL_n(\bar{k}) \mid {}^t h_2 T h_1 = T\}, \\ \mathbb{X}_T(\bar{k}) &= \mathfrak{X}_T(\bar{k}) / \mathbb{U}(T) \supset \mathbb{X}_T(k) = \mathbb{X}_T^\Gamma, \end{aligned}$$

where we can consider the similar Γ -action on $\mathbb{U}(H_n)$ and $\mathbb{U}(T)$, since H_n and T are Γ -invariant. We identify

$$\begin{aligned} \mathbb{U}(H_n)^\Gamma &= \{(g, \bar{g}) \in GL_{2n}(k') \times GL_{2n}(k') \mid {}^t \bar{g} H_n g = H_n\} \quad \text{with } U(H_n), \\ \mathbb{U}(T)^\Gamma &= \{(h, \bar{h}) \in GL_n(k') \times GL_n(k') \mid {}^t \bar{h} T h = T\} \quad \text{with } U(T), \end{aligned}$$

where and henceforth we write \bar{g} instead of g^τ for a matrix g with entries in k' .

LEMMA C.1. *The map*

$$\varphi_T : X_T \longrightarrow \mathbb{X}_T(k), \quad xU(T) \longmapsto (x, \bar{x})\mathbb{U}(T)$$

is injective.

Proof. Assume $\varphi_T(xU(T)) = \varphi_T(yU(T))$. Then, for some $(h_1, h_2) \in \mathbb{U}(T)$, we have $y = xh_1$, $\bar{y} = xh_2$. Taking n linearly independent rows from x , we make $x_0 \in GL_n(k')$. Then $y_0 = x_0 h_1 \in$

$GL_n(k')$, $h_1 = x_0^{-1}y_0 \in GL_n(k')$ and $h_2 = \overline{x_0^{-1}y_0} = \overline{h_1} \in GL_n(k')$. Hence $(h_1, h_2) \in U(T)$ and $xU(T) = yU(T)$. \square

Hereafter we understand X_T as a subspace of $\mathbb{X}_T(k)$ through φ_T . Set

$$T_1 = \begin{pmatrix} \pi & 0 \\ 0 & 1_{n-1} \end{pmatrix}, \quad \widetilde{\eta}_\pi = (\eta_\pi, \eta_\pi), \quad \eta_\pi = \begin{pmatrix} \sqrt{\pi}^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix}.$$

LEMMA C.2. *The map*

$$f : \mathbb{X}_{T_1}(\overline{k}) \longrightarrow \mathbb{X}_{1_n}(\overline{k}), \quad (x, y)\mathbb{U}(T_1) \longmapsto (x\eta_\pi, y\eta_\pi)\mathbb{U}(1_n)$$

is well defined and sends $\mathbb{X}_{T_1}(k)$ into $\mathbb{X}_{1_n}(k)$.

Proof. For any $(x, y) \in \mathfrak{X}_{T_1}(\overline{k})$ and $\widetilde{h} = (h_1, h_2) \in \mathbb{U}(T_1)$, we have

$$(xh_1\eta_\pi, yh_2\eta_\pi) = (x\eta_\pi, y\eta_\pi)\widetilde{\eta}_\pi^{-1}\widetilde{h}\widetilde{\eta}_\pi \in (x\eta_\pi, y\eta_\pi)\mathbb{U}(1_n),$$

hence f is well defined. Take any $\alpha = (x, y)\mathbb{U}(T_1) \in \mathbb{X}_{T_1}(k)$. For each $\sigma \in \Gamma$, there exists some $h_\sigma \in \mathbb{U}(T_1)$ satisfying $\sigma(x, y) = (x, y)h_\sigma$, and $\sigma(\sqrt{\pi}) = \pm\sqrt{\pi}$. Hence

$$\begin{aligned} \sigma(f(\alpha)) &= (x, y)h_\sigma\widetilde{\eta}_\pi \left(\begin{pmatrix} \pm 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} \right) \mathbb{U}(1_n) \\ &= (x\eta_\pi, y\eta_\pi)\widetilde{\eta}_\pi^{-1}h_\sigma\widetilde{\eta}_\pi \mathbb{U}(1_n) = (x\eta_\pi, y\eta_\pi)\mathbb{U}(1_n) = f(\alpha) \end{aligned}$$

Hence $\sigma f = f$ for any $\sigma \in \Gamma$, and f sends the set $\mathbb{X}_{T_1}(k)$ of k -rational points into to the set $\mathbb{X}_{1_n}(k)$. \square

LEMMA C.3. *The set $\mathbb{X}_{1_n}(k)$ contains at least two $U(H_n)$ -orbits, X_{1_n} and $f(X_{T_1})$.*

Proof. By Lemma C.2, we see

$$\begin{array}{ccccc} X_{T_1} & \subset & \mathbb{X}_{T_1}(k) & \subset & \mathbb{X}_{T_1}(\overline{k}) \\ & & \downarrow f & & \downarrow f \\ X_{1_n} & \subset & \mathbb{X}_{1_n}(k) & \subset & \mathbb{X}_{1_n}(\overline{k}), \end{array} \tag{C.1}$$

hence it suffices to show $X_{1_n} \cap f(X_{T_1}) = \emptyset$. Assume there exists some $(x, \overline{x}) \in \mathfrak{X}_{T_1}$, $(y, \overline{y}) \in \mathfrak{X}_{1_n}$, $(h_1, h_2) \in U(1_n)$ such that $(x\eta_\pi, \overline{x}\eta_\pi) = (y, \overline{y})(h_1, h_2)$. Taking suitable linearly independent rows from x , we have $x_0 \in GL_n(k')$ and $y_0 = x_0\eta_\pi h_1^{-1} \in GL_n(k')$. Then we see

$$h_1\eta_\pi^{-1} = y_0^{-1}x_0 \in GL_n(k'), \quad \overline{h_1\eta_\pi^{-1}} = h_2\eta_\pi^{-1}.$$

Setting $h = h_1\eta_\pi^{-1}$, we have

$${}^t\overline{h}1_nh = \eta_\pi^{-1}{}^th_21_nh_1\eta_\pi^{-1} = \eta_\pi 1_n \eta_\pi = T_1,$$

which is impossible for $h \in GL_n(k')$. \square

We consider

$$\mathbb{N} = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & k_1 \end{pmatrix}, \begin{pmatrix} h_2 & 0 \\ 0 & k_2 \end{pmatrix} \right) \mid (h_1, h_2), (k_1, k_2) \in \mathbb{U}(1_n) \right\}$$

and the similar Γ -action on \mathbb{N} as before. Then we may identify \mathbb{N}^Γ with

$$N = \left\{ \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} \mid h, k \in U(1_n) \right\}.$$

Setting

$$y_0 = \begin{pmatrix} \xi 1_n \\ 1_n \end{pmatrix} \in \mathfrak{X}_{1_n}, \quad \xi = \frac{1 + \sqrt{\varepsilon}}{2},$$

we see the stabilizer in $U(H_n)$ at $y_0 U(1_n) \in X_{1_n}$ is given by (cf. [Hir11, Lemma 1.1])

$$\nu N \nu^{-1}, \quad \nu = \begin{pmatrix} \xi 1_n & \bar{\xi} 1_n \\ 1_n & -1_n \end{pmatrix} \in GL_{2n}(\mathcal{O}_{k'}).$$

On the other hand, as is noted in (1.12) the stabilizer in $G = U(j_{2n})$ at 1_{2n} is given by

$$\mu N \mu^{-1}, \quad \mu = \begin{pmatrix} 1_n & 1_n \\ j_n & -j_n \end{pmatrix} \in GL_{2n}(k).$$

Since

$$\begin{aligned} H_n[\nu] &= \begin{pmatrix} \bar{\xi} 1_n & 1_n \\ \xi 1_n & -1_n \end{pmatrix} \begin{pmatrix} 1_n & -1_n \\ \xi 1_n & \bar{\xi} 1_n \end{pmatrix} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \\ j_{2n}[\mu] &= \begin{pmatrix} 1_n & j_n \\ 1_n & -j_n \end{pmatrix} \begin{pmatrix} 1_n & -1_n \\ j_n & j_n \end{pmatrix} = 2 \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \end{aligned}$$

we have

$$\mu \nu^{-1} U(H_n) \nu \mu^{-1} = G, \quad \mu \nu^{-1} \mathbb{U}(H_n) \nu \mu^{-1} = G(\bar{k}). \quad (\text{C.2})$$

where we identify ν and μ with their images in $R_{k'/k}(GL_{2n})$. Thus we have

$$\begin{aligned} \mathbb{X}_{1_n}(\bar{k}) &\cong \mathbb{U}(H_n) / \nu N \nu^{-1} \xrightarrow{\varphi} G(\bar{k}) / \mu N \mu^{-1} \cong X_n(\bar{k}) \\ \cup & \qquad \qquad \qquad \cup \\ \mathbb{X}_{1_n}(k) \supset X_{1_n} &\cong U(H_n) / \nu N \nu^{-1} \xrightarrow{\varphi} G / \mu N \mu^{-1} \cong G \cdot 1_{2n} \subset X_n, \end{aligned} \quad (\text{C.3})$$

where φ is the conjugation determined by (C.2). Then we have the following by the commutative diagram (C.3), Lemma C.3, and Theorem 1.9.

PROPOSITION C.4. *The above φ gives an isomorphism between the sets of k -rational points*

$$U(H_n) \backslash \mathbb{X}_{1_n}(k) \cong G \backslash X_n,$$

and $U(H_n)$ -orbit decomposition

$$\mathbb{X}_{1_n}(k) = X_{1_n} \sqcup f(X_{T_1}); \quad X_{1_n} \cong G \cdot x_0, \quad f(X_{T_1}) \cong G \cdot x_1,$$

where x_0 and x_1 are the representatives of G -orbits in X_n given in Theorem 1.9.

Now we assume that q is odd and consider the Cartan decomposition of $X_T = \mathfrak{X}_T / U(T)$. Set $K' = U(H_n) \cap GL_{2n}(\mathcal{O}_{k'})$ and recall $K = G \cap GL_{2n}(\mathcal{O}_{k'})$. Since $2 \notin (\pi)$, we see $\mu \in GL_{2n}(\mathcal{O}_{k'})$ and

$$\mu \nu^{-1} K' \nu \mu^{-1} = K.$$

Hence we have the bijective correspondence

$$K' \backslash \mathbb{X}_{1_n}(k) \longleftrightarrow K \backslash X_n,$$

and we see the space X_T inherits the Cartan decomposition of X_n , the space of unitary hermitian space. Thus we have the following.

THEOREM C.5. Assume k has odd residual characteristic and take any $T \in \mathcal{H}_n(k')$. Then

$$\mathfrak{X}_T = \bigsqcup_{\substack{\lambda \in \Lambda^+ \\ \lambda \sim T}} K' x_\lambda h_\lambda U(T),$$

where

$$x_\lambda = \begin{pmatrix} \xi \pi^\lambda \\ 1_n \end{pmatrix} \in \mathfrak{X}_{\pi^\lambda},$$

$\lambda \sim T$ means that $|\lambda| \equiv v_\pi(\det(T)) \pmod{2}$ and guarantees the existence of $h_\lambda \in GL_n(k')$ satisfying $T = \pi^\lambda[h_\lambda]$.

The above decomposition has been expected in [Hir11, Remark 4.2]. In [Hir11], where we have known the disjointness of orbits in the right hand side by explicit formulas of spherical functions $\omega_T(y; z)$, but we didn't know they are enough. By Theorem C.5, we see the spherical Fourier transform F_T is isomorphic in [Hir11, Theorem 4.1] if q is odd.

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